Utilizing Technology to Facilitate the Transition from Secondary- to Tertiary Level Linear Algebra

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Abstract

A common perception among researchers (e.g. [Dorier, 2000]) in mathematics education is that the transition between secondary- and tertiary level of mathematics may be problematic for the students. In particular, the exact and abstract nature of the theory of Linear algebra versus its arithmetic-geometric presentation in school appears to be difficult for the novice students. The application of properties for defining concepts at university in contrast to their usage for describing concepts in school points out a possible occurrence of obstacles for learning ([Donevska-Todorova, 2016b]) and discrepancies in procedural and conceptual understanding ([Donevska-Todorova, 2016a]).

The aim of this study is to examine how could upper-high school students develop a conceptual understanding based on concept definition and concept image ([Tall & Vinner, 1981]) in connection to multiple modes of description and thinking ([Hillel, 2000]; [Sierpinska, 2000]) about concepts such as bi-linearity exemplified by the dot product of vectors and multi-linearity exemplified by determinants. In order to achieve this, I have created a teaching/ learning sequence in a dynamic geometry environment (DGE), then implemented it and evaluated it in a high school in Berlin, following a complete cycle of design-based research ([The Design-Based Research Collective, 2003]; [Kelly, Lesh & Baek, 2008]) and conducting a multiple-level data analysis.

The findings of the study show not only that widening students' concept images, developing multiple modes of thinking and gaining deeper conceptual understanding can successfully be mediated by dynamic geometries, but also give insights into an eventual theoretical model of how can they be further examined (using guiding features [Donevska-Todorova, 2015]). Moreover, the study promotes authorized open-source interactive teaching/ learning materials for further sustainable practice and research. It opens new research questions about revisiting axiomatic approaches on local levels ([Freudenthal, 1971]) in upper high-school Linear algebra which may base on the integration of all three modes of description and thinking geometric, algebraic and abstract possibly facilitated by DGEs.

Key words: Linear algebra, transition, upper high school, conceptual understanding, concept definition, concept image, modes of description and thinking, axiomatic, design-based research, dynamic geometry

Zusammenfassung

Es ist eine weit verbreitete Wahrnehmung, dass der Übergang zwischen der Mathematik der gymnasialen Oberstufe und der Mathematik an der Universität für Studierende problematisch sein kann (siehe u. a. [Dorier, 2000]). Besondere Verständnisschwierigkeiten in Bereich der lineare Algebra bereiten den Studierenden die verschiedenen Herangehensweisen auf diesen beiden Ebenen. Dies lässt sich auf die strukturell-axiomatischer Herangehensweisen an die lineare Algebra an der Universität, im Gegensatz zu ihrer arithmetisch-geometrischen Darstellung in der Schule, zurückführen. Während z. B. Rechenregeln in der Schule als Eigenschaften auftreten, dienen sie an der Universität der (axiomatischen) Definition von Strukturen. Dies bedingt ebenfalls Unterschiede im prozeduralen und konzeptuellen Verständnis ([Donevska-Todorova, 2016a]).

Ziel dieser Arbeit ist es, zu untersuchen, wie Schüler der gymnasialen Oberstufe konzeptuelles Verständnis (Bezug nehmend auf die Theorien von concept definition und concept image, [Tall & Vinner, 1981]) in Verbindung mit multiplen Modi der Beschreibung und des Denkens ([Hillel, 2000]; [Sierpinska, 2000]), von Konzepten wie Bilinearität (am Beispiel des Skalarprodukts) und Multilinearität (am Beispiel von Determinanten) gewinnen können. Um dies zu erreichen, wurde eine Lehr-/Lernsequenz unter Verwendung einer dynamischen Geometriesoftware entwickelt. Die Lerneinheit wurde an einem Berliner Gymnasium eingesetzt und dabei ein vollständiger design-based research Zyklus ([The Design-Based Research Collective, 2003]; [Kelly, Lesh & Baek, 2008]) durchlaufen und eine multiple-level Datenanalyse durchgeführt.

Die Ergebnisse der Untersuchung zeigen nicht nur, dass eine Erweiterung der Vorstellungen der Schüler, eine Entwicklung multipler Denkmodi und ein Gewinn tieferen konzeptuellen Verständnisses in der linearen Algebra erfolgreich vermittelt werden können, sondern geben auch Einblicke in ein mögliches theoretisches Modell, mit dessen Hilfe sich diese Prozesse weiter untersuchen lassen (basierend auf fünf Leitlinien, [Donevska-Todorova, 2015]). Weiterhin werden die interaktiven Lehr-/Lernmaterialien für die weitere Verwendung im Rahmen von Lehre und Forschung zur Verfügung gestellt. Es öffnen sich neue Forschungsfragen hinsichtlich lokalen Axiomatisierens ([Freudenthal, 1971]) in der linearen Algebra der gymnasialen Oberstufe, welches auf einer Integration geometrischer, algebraischer und strukturell-axiomatischer Beschreibungsmodi und Denkweisen, unterstützt durch dynamische

Geometrie Software, basieren könnte.

Stichworte: Lineare Algebra, Übergänge, gymnasiale Oberstufe, konzeptuelles Verständnis, concept definition, concept image, Repräsentationsmodi und Denkmodi, design-based research, Technologie, dynamische Geometrie

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Ana Donevska-Todorova

Contents

Introduction

Using the theory of [Tall & Vinner, 1981] about concept definition and concept image, I was interested to know whether students know more definitions for a single concept and what kind of images do they develop about concepts in Linear algebra. My further interest was on the following questions. Are students familiar with the fact that such definitions may be given in multiple modes of description and thinking, according to the theories of [Hillel, 2000] and [Sierpinska, 2000]? Moreover, can students establish connections between geometric, algebraic and axiomatic-structural modes of thinking and translate from one into another? Can I detect and formulate their difficulties using these theories? Can I design a learning environment which may be helpful for overcoming their difficulties and promoting integration of multiple modes of description and thinking and therefore, widening concept images and deepening conceptual understanding? Can it be a Dynamic Geometry Environment (DGE) in which students can experience the embodied, symbolic-proceptual and formal-axiomatic worlds of [Tall, 2003]? Can my findings stimulate further research about defining conceptual understanding on the basis of concept definitions, concept images and multiple modes of description and thinking? Finally, can my design-based research [The Design-Based Research Collective, 2003] contribute in sustainable research by proposing possible shifts in the learning trajectories about local axiomatic [Freudenthal, 1971] from tertiary to secondary Linear algebra and by suggesting a minor reconsideration of the completely abandoned axiomatic approaches in school mathematics during the "New Math" era?

In order to come up with answers on these questions I undertook a complete design-based research cycle consisted of seven phases and I report about it through six chapters of this thesis.

Chapter 1 considers epistemological and historical questions related to the evolution of concepts in the theory of Linear algebra. The focus of the analysis is on the historical advancement of the concepts as vectors, dot and cross products of vectors, linearity, matrices, and determinants. The epistemological and historical analysis suggests different didactical ideas regarding arithmetic-algebraic, geometric and axiomatic approaches to the teaching and learning of the mentioned concepts. A part of these ideas have been used for identifying students' difficulties in learning, in the third chapter, or for creating the artifact, in the fourth chapter of this thesis.

Chapter 2 sets the theoretical framework of the undertaken research. On the beginning, I give an overview to relevant scientific literature defining conceptual understanding in mathematics education and suggest five features of conceptual understanding. In order to investigate how students develop it according to these features, I refer to the theories of [Tall & Vinner, 1981] about concept definition and concept image with an emphasis to the theories about multiple modes of description ([Hillel, 2000]) and thinking ([Sierpinska, 2000]) which are specific for Linear algebra. I suggest that the combination of these theories is suitable for investigating the development of students' deeper understanding of the dot product and determinants. I see a potential in supporting such development with the use of a dynamic geometry environment (DGE) which is due to its characteristic for simultaneous dynamism of multiple modes of description.

Further on, in **Chapter 3** I try to illustrate the research problem about students' difficulties in learning the dot product of vectors and determinants. Afterward, I state the research questions and the methodology and explain why have I chosen the design-based-research as an adequate methodology in order to try to answer the posed research questions in my study. Here, I refer to an additional theoretical framework concerning an instrumental genesis of complex classroom phenomena in DGE ([Drijvers et al., 2010]), which serves for multiple levels data analysis through the phases of the design cycle.

In the **4th Chapter** I show the artifact which I have created, in a form of a hypothetical learning trajectory (HLT). It consists of teaching and learning materials based on interactive applets, which aim to help students in understanding the concepts of bi-linearity, as exemplified by the dot product and multi-linearity, as exemplified by the determinants. Moreover, I emphasize the presence of all three specific modes of description and thinking in these materials.

In **Chapter 5** I show the findings of my doctoral project gained during the actual learning trajectory (ALT) in an upper high school (Gymnasium) in Berlin. I discuss students' performance in the created DGE, by showing their practical engagements, oral and written outputs in both, technology- and paper-pencil-based tasks. I finish this chapter with a discussion of assessment, evaluation, and dissemination of the results of the study.

I summarize my work in the last **Chapter 6** where I derive conclusions upon the obtained results, explain the contribution of this study and give an outlook for possible teaching, further design and research.

Chapter 1

Epistemological, Historical and Didactical Considerations about Concepts in Linear Algebra

There is no doubt about the fact that the history of mathematics presents a rich source for research in mathematics education ([Ho, 2008];[Krantz, 2006];[ICMI, 2000]). It has the potential to shape a variety of didactic situations which combined with a modern context may lead to a substantial contribution to teaching and learning mathematics. In particular, a closer look at the historical development of Linear algebra may offer ideas regarding curriculum genesis or may influence the instruction and instructional design of the subject ([Larson, 2010]). Important didactical questions arising from historical contexts which are to be tackled in this chapter are the following.

(1) How were Linear algebra concepts defined in history and how did *concept definitions*[1] evolve throughout a long period?

(2) What kind of *concept representations*[2] have been used in the theory of Linear algebra and how did they influence the teaching and learning of the subject?

[1]A discussion related to the meaning of the term "concept definition" is given in Section 2.2

[2]Concept representations in this doctoral study relate to concepts in Linear algebra. They are thoroughly discussed in Section 2.3 of this thesis, when they are assigned to multiple modes of description and thinking within a particular theoretical framework for research in the teaching and learning of Linear algebra.

> How firstly developed representations such as *geometric* and *arithmetic* representations, lead towards the emergence of the *axiomatic-structural* representations after a long period of evolution and how do they impact instruction even today? To which *other concepts* have the concepts in concern been *connected* depending on their representations?

This part of the study focuses on the development of concepts of vectors, the dot product of vectors and determinants, their definitions and representations. Namely, the next sections try to offer some answers to the above questions, starting with determinants in Section 1.1 and continuing with vectors in Section 1.2. Then, Section 1.3 focuses on the influence of the historical genesis of the concepts, their definitions and representations on the teaching approaches in school Linear algebra. Section 1.4 offers a closer insight into the axiomatic-structural approaches and generalizations, which are specific for university Linear algebra, having historical facts as a background. In addition, this section comments local levels which may facilitate the transition between the secondary- and tertiary level Linear algebra.

1.1 Arithmetic-algebraic Approaches and the Birth of Determinants

Comparing national Linear algebra curricula or trying to configure a suitable one according to certain contextual needs, there is a possibility to recall historical facts. For example:

> [...] the typical structuring of a Linear algebra course follows the historical order of development more closely than that of many other courses studied by the undergraduates. For example, the first Linear algebra topic usually encountered is that of solutions of systems of linear equations, a subject which to some extent was studied by the Babylonians nearly 4000 years ago. The next topics may well be the determinants, which date from roughly 300 years ago, and the elements of vector geometry in 2-space and 3-space, a concern of the early nineteenth century. The more abstract notations of vector spaces and linear transformations, built upon concrete foundations, were not fully developed in the mathematical literature until the late nineteenth and early twentieth century and are typically studied toward the end of a Linear algebra program ([Katz, 1995], p. 189).

The above-suggested curriculum order of having determinants at the beginning of a Linear algebra course, without a prior study on matrices, follows the historical development, although it is a non-typical order today. This suggestion shows the importance of the posed questions above. Namely, if determinants were historically born prior matrices, (1) how were they defined and (2) in connection with which other concepts were they represented?

> The history of determinants is an unusually interesting part of the history of elementary mathematics in view of the fact that it illustrates very clearly some of the difficulties in this history which result from the use of technical terms therein without exhibiting the definitive meaning which is to be given to these terms. Many modern writers have based their definitions of a determinant on the existence of a square matrix. [...] From this point of view a determinant does not exist without its square matrix, and, judging from many textbooks on elementary mathematics, it is likely that many students consider the square matrix as an essential part of a determinant, so that the term determinant conveys to them a dual concept composed of a squared matrix and a certain polynomial associated therewith. When they speak of the rows and columns of a determinant they are naturally thinking of the polynomial implied by the term determinant. When a student who is familiar with no definition of the term determinant except the dual one [...] meets with the common statement that the discovery of determinants is usually ascribed to G. F. Leibniz, he naturally concludes that a square matrix and a polynomial were associated by G. F. Leibniz in about the same way as they are associated at the present time. This is, however, not the case ([Miller, 1930], p. 216).

The dilemma whether determinants precede matrices or vice versa may seem a bit surprising for a modern learner of Linear algebra. The existence of certain historical facts offer some insights. Namely, an example which is pointed out in some literature for example by [Larson, 2010] and [Katz, 1997] is found in Chapter 8 of the most famous book of Chinese mathematics Nine Chapters on the Mathematical Art from 200 BC. It is the following.

Example.

> There are three types of corn, of which three bundles of the first, two of the second, and one of the third make 39 measures. Two of the first, three of the second and one of the third make 34 measures. And one of the first, two of the second and three of the third make 26 measures. How many measures of corn are contained in one bundle of each type?

The problem is written as a table on a counting board in which the coefficients of the system of three linear equations in three unknowns are listed as columns from the right to the left, in contrary of today's down listing the coefficients as rows of a matrix or a determinant. This is, of course, identical because of two properties

of determinants: one, determinants change their signs when two rows (columns) change their places and two, determinants have the same value of a matrix and its transpose.
The undertaken steps towards the solution of the example are given with:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ & & \\ 26 & 34 & 39 \end{array} \qquad \begin{array}{ccc} 0 & 0 & 3 \\ 4 & 5 & 2 \\ 8 & 1 & 1 \\ & & \\ 39 & 24 & 39 \end{array} \qquad \begin{array}{ccc} 0 & 0 & 3 \\ 0 & 5 & 2 \\ 36 & 1 & 1 \\ & & \\ 99 & 24 & 39 \end{array}$$

Figure 1.1: Solution of the Problem with Corn in 200 BC

The author of the problem has given instructions for multiplication of the middle column by 3 and subtraction of the right column as many times as possible which is also done with the right column being subtracted from the first column multiplied by 3. Then, the left column is multiplied by 5 and the middle column is subtracted as many times as possible, so the third scheme is obtained. The final solution is obtained first for the third type of corn, then for the second type and at the end for the first type of corn.
Today, the given example would probably be written with one of the following notations:

$$\begin{cases} 3x + 2y + z = 39 \\ 2x + 3y + z = 34 \\ x + 2y + 3z = 26 \end{cases} \quad \text{or} \quad \begin{bmatrix} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{bmatrix}$$

Any learner of Linear algebra can recognize the Gaussian elimination method as it is called today, in the above example. It shows two historical facts, one, that the Gaussian elimination method existed a long time before Gauss and, two, that it developed directly from the problems with systems of linear equations and not from matrices ([Larson, 2010]). Although research is separated on whether this particular example served as a root for determinants ([Larson, 2010]) or matrices ([Katz, 1997]; [Vogel, 1968]) or both determinants and matrices ([Ershaidat, 2007]; [Brieskorn, 1983]), a deeper look at the history leads to an idea that, no matter how strange to the modern reader it may seem, determinants were born and developed through a long period *in connection with* the desire of finding general rules for the solutions of *systems of linear equations* (e.g. Figure 1.2 and other earlier examples in [Muir, 1905]). This historical fact legitimated the pedagogical hint about the curriculum order suggested by [Katz, 1995].
Deeper investigation in history detects more serious beginnings appearing in the 16th, 17th and the 18th centuries. One of the first trials related to solving systems of two linear equations is the following.

mination ', and opening with the theorem: If there be any number of linear equations involving as many unknown quantities in the form

$$a_1x + b_1y + \dots = A_1,$$
$$a_2x + b_2y + \dots = A_2,$$
$$\dots\dots\dots$$

we shall have

$$x = \frac{\alpha A_1 - \beta A_2 + \gamma A_3 - \dots}{\alpha a_1 - \beta a_2 + \gamma a_3 - \dots};$$

the coefficients α, β, γ, ... being obtained from the original coefficients by exterminating all the unknown quantities except x in succession. This is followed by two pages of elucidation, in which appear the actual expansions of determinants of the second and third order.

Figure 1.2: Determinants in the Work of Young, (1821, p. 2) [5]

> **Cardano**, in *Ars Magna* ([Cardano, 1545]), gives a rule for solving a system of two linear equations which he calls *regula de modo* and which [...] he calls *mother of rules*! This rule gives what essentially is Cramer's rule for solving a 2×2 system although Cardano does not make the final step. Cardano therefore, does not reach the definition of a determinant, but with the advantage of hindsight, we can see that his method does lead to the definition ([Ershaidat, 2007], p. 1-2).

In 1693, but published in 1850, **Leibniz** (1646-1716) had the first attempt to write the solutions of systems of linear equations with a specific notation and to discuss on the terms and the sign, which left traces similar (even though not as efficient as) to those used and promoted later by Cramer ([Muir, 1905]; [Dorier, 2000]).
After **Von Maclaurin** (1698-1746) determined a general solution for systems with three linear equations in three unknowns in 1729 (published post-hum 1748), **Cramer** (1704-1752) in 1750, without knowing the work of Maclaurin, described a rule[3] for solving regular systems of linear equations with two, three and four unknowns ([Wittmann, G., 2003]). The rule used a specific notation and got his name. This notation will later be called *a determinant* for the first time by **Cauchy** (1789-1857) ([Wittmann, G., 2003]; [Dorier, 2000]; [Cajori 2000]).

> The term of determinant was introduced by Cauchy in a memoir presented to the French Academy in the 1812 and published in 1815. This was also the first work with a theoretical approach in which more than merely a rule for calculation was given. Indeed, Cauchy analyses the determinant from the perspective of a function in n variables (taking only two opposite values through a permutation of its variables) ([Dorier, 2000], p. 60).

[3]Cramer makes the calculations for $n = 2$ and $n = 3$, implicitly supposing that the main determinant differs from zero, then he says that this can be easily generalized to greater values of n ([Dorier, 2000], p. 61).

Cauchy's work shows a change in the approaches and the use of representations from typical *arithmetic* (for calculations) into *algebraic* (in terms of functions). These discoveries lead to other numerous studies, among which the studies of **Smith** (1826-1883), **Silvester** (1814-1897) and **Cayley** (1821-1895) made the most remarkable steps forward ([Macfarlane, 1916]). They made a considerable change in the approaches due to the technical difficulties and determinants' inefficiency in dealing with complex calculations. After Silvester first used the term of *a matrix*, Cayley accepted the terminology and wrote *A Memoir on the Theory of Matrices* ([Cayley, 1857], p. 17)(Figure 1.3).

II. *A Memoir on the Theory of Matrices.* *By* Arthur Cayley, *Esq., F.R.S.*

Received December 10, 1857,—Read January 14, 1858.

The term matrix might be used in a more general sense, but in the present memoir I consider only square and rectangular matrices, and the term matrix used without qualification is to be understood as meaning a square matrix; in this restricted sense, a set of quantities arranged in the form of a square, *e. g.*

$$\left(\begin{array}{l} a\ ,\ b\ ,\ c \\ a',\ b',\ c' \\ a'',\ b'',\ c'' \end{array}\right)$$

is said to be a matrix. The notion of such a matrix arises naturally from an abbreviated notation for a set of linear equations, viz. the equations

$$\begin{aligned} X &= ax + by + cz, \\ Y &= a'x + b'y + c'z, \\ Z &= a''x + b''y + c''z, \end{aligned}$$

may be more simply represented by

$$(X, Y, Z) = \left(\begin{array}{l} a\ ,\ b\ ,\ c \\ a',\ b',\ c' \\ a'',\ b'',\ c'' \end{array}\right)(x, y, z),$$

and the consideration of such a system of equations leads to most of the fundamental notions in the theory of matrices. It will be seen that matrices (attending only to those

Figure 1.3: Cayley's Definition and Notation of Matrices ([Cayley, 1857], p. 17)

Cayley's publication shows two important things about the history. First, it is not always clear if authors at this particular time referred to matrices or determinants in their work[4]. It seems that Cayley used the term "matrix" to refer to a determinant and "square or rectangular matrix" to refer to a matrix (Figure 1.3). However, the idea of distinguishing between two different concepts, a determinant, and a matrix, was now born. Second, it shows new notations for representing systems of linear equations using matrices (Figure 1.3). Further on, Cayley's work directly stimulated the birth of linear transformations ([Salmon, 1859]) and opened a new door in the history of Linear algebra in connection with Geometry.

[4]Some mathematicians used the term determinant to refer to both determinant and matrix in today's meaning. For an example see ([Salmon, 1859], p. 3).

The *Theory of Determinants* was in expansion in the 19th century. In this period, mathematicians tried to define determinants without the use of systems of linear equations. For example, a definition of determinants in order n based on permutations is the following:

> **2. Definition.** Unter der Determinanten des Systems von n^2 Elementen, welche in n Reihen von je n Elementen stehen und von denen das $k$te der $i$ten Reihe durch $a_{i,k}$ bezeichnet wird, versteht man das Aggregat der Produkte von je n solchen Elementen, die sämtlich aus verschiedenen Zeilen und Kolonnen entnommen sind. Das Anfangsglied der Determinante ist das Produkt der Elemente der Diagonalreihe $a_{1,1}a_{2,2}...a_{n,n}$. Aus dem Anfangsglied werden die übrigen Glieder abgeleitet, indem man die ersten Numern unverändert lässt und die zweiten permutiert. Die einzelnen Glieder werden positiv oder negativ genommen, je nachdem die Permutationen der Nummern, durch welche sie entstanden sind, derselben Klasse angehören als die erste Complexion der Nummern, oder nicht. Die Determinante von n^2 Elementen heißt n-ten Grades, weil ihre Glieder Produkte von n Elementen sind ([Baltzer, 1864], p. 6).

An example of determinants in order 2, 3 and 4 follows after this definition (Figure 1.4).

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \cdot\cdot & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdot\cdot & a_{2,n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n,1} & a_{n,2} & \cdot\cdot & a_{n,n} \end{vmatrix} = \Sigma \pm a_{1,1}\ a_{2,2} \cdot\cdot\ a_{n,n} = \frac{1 \mid 2 \mid \cdot\cdot \mid n}{1 \mid 2 \mid \cdot\cdot \mid n} = \begin{pmatrix} 1 & 2 & \cdot\cdot & n \\ 1 & 2 & \cdot\cdot & n \end{pmatrix}.$$

Beispiele.

$$\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = ab_1 - a_1 b.$$

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = ab_1c_2 - ab_2c_1 + a_1b_2c - a_1bc_2 + a_2bc_1 - a_2b_1c.$$

$$\begin{vmatrix} a & b & c & d \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = \begin{matrix} ab_1c_2d_3 - ab_1c_3d_2 + ab_2c_3d_1 - ab_2c_1d_3 + ab_3c_1d_2 - ab_3c_2d_1 \\ -a_1bc_2d_3 + a_1bc_3d_2 + a_1b_2cd_3 - a_1b_2c_3d - a_1b_3cd_2 + a_1b_3c_2d \\ +a_2bc_1d_3 - a_2bc_3d_1 - a_2b_1cd_3 + a_2b_1c_3d + a_2b_3cd_1 - a_2b_3c_1d \\ -a_3bc_1d_2 + a_3bc_2d_1 + a_3b_1cd_2 - a_3b_1c_2d - a_3b_2cd_1 + a_3b_2c_1d. \end{matrix}$$

Figure 1.4: Definition of a Determinant with n^2 Elements ([Baltzer, 1864], p. 7)

As an application of determinants ([Baltzer, 1864], p. 175) writes about the oriented area of a triangle and volume of a tetrahedron (Figure 1.5).

This part of the historical analysis offers some answers to the previously stated questions (1) and (2) in the introduction of this Chapter 1, on p. 3. First, it shows how *concept definitions* of determinants evolved from primarily being exclusively dependent on systems of linear equations resulting with the Cramer's rule, to being

§. 15. Die Dreiecksfläche und das Tetraedervolum.

1. Wenn O den Anfang beliebiger coordinirter Axen bedeutet, wenn x, y und x_1, y_2 die mit den Axen parallelen Coordinaten der Punkte A und B sind und die Geraden, auf denen OA und OB liegen, wie die Strecken selbst durch r, r_1 bezeichnet werden, wenn ferner die Dreiecksfläche OAB positiv oder negativ genommen wird, je nachdem der Sinn der Drehung, welcher durch die Ordnung der Punkte O, A, B bestimmt ist, mit dem positiven Sinn der Ebene, in welchem positive Winkel derselben beschrieben werden, übereinstimmt oder nicht, so ist *)

$$2\,OAB = r r_1 \sin r r_1 = \begin{vmatrix} x & y \\ x_1 & y_1 \end{vmatrix} \sin xy\,.$$

Figure 1.5: Application of Determinants for Areas of Triangles ([Baltzer, 1864], p. 175)

connected with permutations without any dependence on systems of linear equations. Moreover, the process of generalization from lower towards higher dimensions (Figure 1.4) (firstly, only systems of linear equations with two/three equations in two/ three unknowns) was now substituted with the opposite one, from more general towards more specific. This illustrates the change from inductive towards deductive approaches, an important didactical matter. Second, determinants' representations have mainly been *arithmetic-algebraic* for a long time and the notations improved along with the concept's development (Figures 1.1, 1.2, 1.3 and 1.4). Further development lead to a later emergence of their applications in geometry, although visual *geometric* representations are rare (Figure 1.5). Third, these geometrical applications show new connections of determinants with the concepts of an oriented area of triangles and volume of tetrahedrons, which have pedagogical importance serving as meaningful resources for learning.

Further analysis of the epistemological genesis of Linear algebra shows crucial growth not only in the concept definition of determinants but also in their representations and notations with double indexing which was not the case before. It can be illustrated by the definition presented in the following Figure 1.6 dating from 1859 (in comparison with the previous one on Figure 1.5 dating from 1864).

This statement "a determinant is only a function of its constituents" ([Salmon, 1859], p. 6) on Figure 1.6 points out that a determinant represents a function of its entries ("constituents"), a similar consideration as by Cauchy, which is a completely new, much closer to the modern definition. Determinants considered as functions receive more *algebraic* rather than pure *arithmetic* flavour. Moreover, this step forward in the idea of defining determinants leads to other important discoveries. Namely, aiming to obtain the multiplicative property of determinants and "more intelligible" general theory [Salmon, 1859], p. 6 states 21 properties of determinants. For an illustration, only seven of these properties are listed below. The reason is avoiding

10. We are now in a position to replace our former definition of a determinant by another, which we make the foundation of the subsequent theory. In fact, since a determinant is only a function of its constituents a_1, b_1, c_1, &c., and does not contain the variables x, y, z, &c., it is obviously preferable to give a definition which does not introduce any mention of equations between these quantities x, y, z.

* Let there be n^2 quantities arrayed in a square of n columns and n rows, then the determinant of these quantities is the sum with proper signs (as explained, Art. 9) of all possible products of n constituents, one constituent being taken from each horizontal and each vertical row. It is very common to write the constituents of a determinant with a double suffix, the first suffix denoting the row, and the second the column, to which the constituent belongs. Thus the determinant of the third order would be written—

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3} \\ a_{2,1}, & a_{2,2}, & a_{2,3} \\ a_{3,1}, & a_{3,2}, & a_{3,3} \end{vmatrix}$$

or else

$$\Sigma \pm a_{1,1},\ a_{2,2},\ a_{3,3},$$

where in the sum the suffixes are interchanged in all possible ways.

Figure 1.6: "The Ultimate Definition" in the Textbook from Salmon ([Salmon, 1859], p. 6)

repetition of analogical properties for determinants in order 2 and 3, which are stated as separate properties by Salmon (1859), but are considered as special cases of one same property today.

> 2. The value of the determinant is not altered if we write the horizontal rows vertically, and vice versa.
>
> 5. The determinant is not altered by writing the horizontal rows vertically, and vice versa; a property which will be proved to be true for every determinant.
>
> 11. The value of the determinant is not altered if the vertical rows be written horizontally, and vice versa.
>
> 12. Any two rows (or two columns) be interchanged, the sign of the determinant is altered.
>
> 13. If two rows (or if two columns) be identical, the determinant vanishes.
>
> 14. If every constituent in any row (or in any column) be multiplied by the same factor, then the determinant is multiplied by that factor.
>
> 15. If every constituent in any row (or in any column) is resolvable into the sum of two others, the determinant is resolvable into the sum of two others ([Salmon, 1859], p. 7-14).

A brief analysis of the above properties leads to the following conclusions. Properties 2, 5 and 11 actually represent one property, which in today's language is: a determinant of the transpose matrix equals the determinant of the given matrix. Property 14 is the homogeneity property and property 15 is the additive property of determinants. These last two properties form the multi-linearity property of determinants. Although there are some misconceptions appearing in the Salmon's properties, as for example properties 3 and 6 which refer to determinants[5] instead of rectangular matrices of systems of two/ three linear equations in three/ four unknowns, respectively, the contribution is of a great importance. It shows a valuable base towards an establishment of the axiomatic approaches. Properties 14 and 15 form the multi-linearity axiom, one of the properties 12 or 13 could be the second axiom which, together with the property of the determinant of the identity matrix, forms an *axiomatic* definition of determinants. Another value is that through analogies for determinants in order 2 and 3, the complete list of the properties shows attempts for a foundation of a general theory of Linear algebra. These are early trials towards the *axiomatic-structural* definitions and representations of concepts in Linear algebra.

In 1864 **Weierstrass** (1815-1897) defined determinants as multi-linear forms, or as functions of n^2 independent variables which satisfy certain properties ([Wittmann, G., 2003]). Ten years later, **Frobenius** (1849-1917) formulated today's *axiomatic* definition of determinants as linear mappings from the ring of $n \times n$ matrices to the associated field which is linear in every row, anti-commutative and normed ([Frobenius, 1905], p. 179). This definition impacts today's systematic of Linear algebra having matrices prior determinants, in contrary of the historical development. In 1875 Frobenius (1849-1917) introduced the concept of rank ([Dorier, 2000]; [Wittmann, G., 2003]) and its efficiency later eliminated the difficulties with solving some problems with determinants.

In conclusion of this section, I organize the above historical data regarding the evolution of determinants in Table 1.1.

It shows that it took centuries of extensive, hard scientific work to achieve something that seems so obvious today. Regarding the first two stated questions (1) and (2) in the introduction of this Chapter 1, p. 3, the following conclusion may be derived. The birth of determinants in an almost pure *arithmetic* context of solutions of systems of linear equations, developed through their definition as a sum of permutations, towards an *algebraic* definition considering them as functions with certain properties. It seems that the history of determinants exemplifies the progress from *arithmetic-algebraic* approaches with a bit of geometric applications towards the *axiomatic-structural* approach in Linear algebra.

[5] Determinants are defined only for squared matrices, but some authors, as was also previously mentioned on p. 11, used the term determinant for both determinant and matrix in today's meaning. For e.g. a typical notation for determinants used at that time was $\begin{Vmatrix} a_1, & a_2, & a_3, \\ b_1 & b_2 & b_3 \end{Vmatrix}$.

Year	Mathemat.	Contribution
1874	Frobenius	Determinant as a linear mapping from the ring of $n \times n$ matrices to a field which is linear in every row, anti-commutative and normed ([Frobenius, 1905], p. 179)
1864	Weierstrass	Defines Determinant as a multi-linear form ([Wittmann, G., 2003])
1864	Baltzer	Definition of Determinant in order n using permutations with examples for $n = 2, 3, 4$ (Figure 1.4). Determinants as oriented areas (Figure 1.5)
1859	Salmon	Defines Determinant as a function of "its constituents" (Figure 1.6) Gives Properties of determinants
1857	Cayley	Clear distinction between Matrix and Determinant (Figure 1.3) Stimulated the birth of linear transformations ([Salmon, 1859])
1856	Silvester	First use of the term "Matrix" ([Macfarlane, 1916])
1812	Cauchy	First use of the term "Determinant" ([Dorier, 2000])
1750	Cramer	Rule for solving SLE in 2,3 and 4 unknowns ([Wittmann, G., 2003])
1729	Maclaurin	General solution for SLE with 3 unknowns ([Wittmann, G., 2003])
1693	Leibniz	Attempt for solving SLE with specific notation and in term of the sign
1545	Cardano	Rule for solving SLE of 2 unknowns ([Ershaidat, 2007])

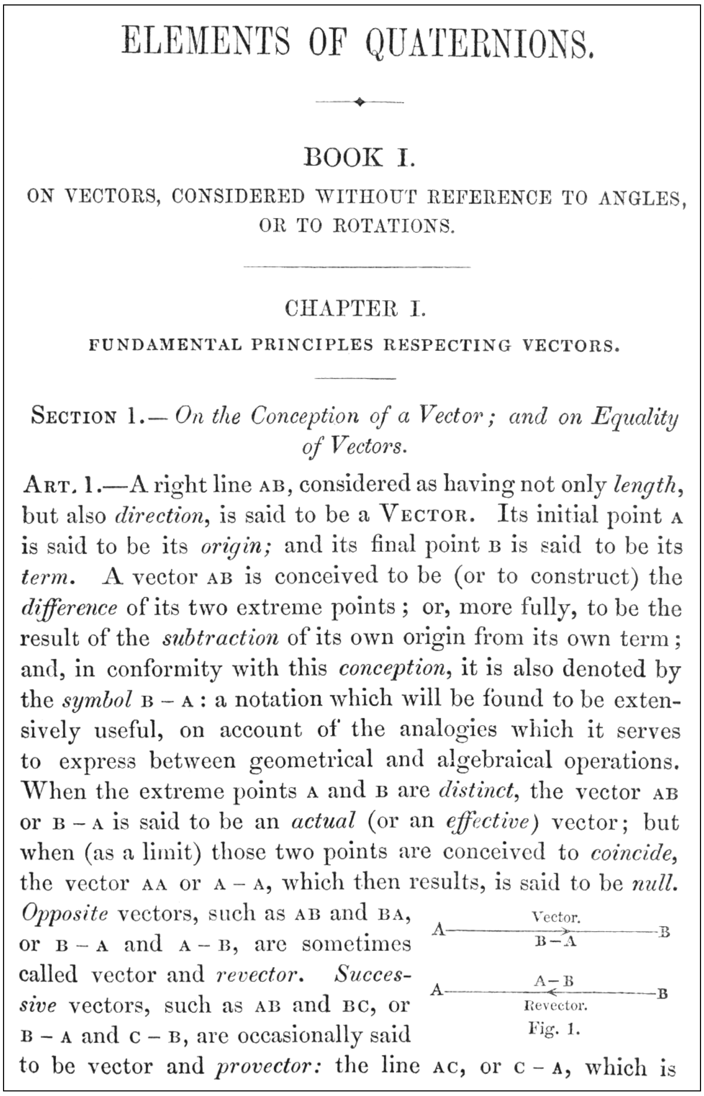
Table 1.1: Historical Development of Determinants

The next Section 1.2 considers the same two questions, but in relation to concepts of vectors, including the dot product and the cross product of vectors.

1.2 Geometric Approaches for Vectors, Dot and Cross Product

Parallel to the arithmetic - algebraic approaches, geometry also developed own approaches towards the same or similar problems in Linear algebra. A period of exchanging algebraic ideas into geometry and vice versa was particularly marked by the name of **Descartes** (1596 - 1650) and later by **Leibniz** (1646 - 1716), who tried to develop geometric calculus in opposition to Analytic geometry ([Gueudet-Chartier, 2004b]). It was in the 17th century when physicists, e. g. **Newton** (1642 - 1727), distinguished *scalar entities* (time, temperature, weight) from *vector entities* (velocity, acceleration, force), yet without using the concepts of vectors. In his work, published in 1831, **Gauss** (1777 - 1855) described imaginary quantities by introducing *geometric representations* of complex numbers as points on a plane ([Wittmann, G., 2003]). With the primary aim of finding a geometrical tool which may help both mathematicians and physicists, **Möbius** (1790 - 1868) introduced the term "directed line-segment" and its notation with the letters of the alphabet. He also defined sums of two collinear and non-collinear directed line-segments, which initiated discoveries in projective geometry afterwards ([Dorier, 2000]).
These historical data point out first trials to define vectors arising from the needs in physics and geometry. Thus, regarding the first questions (1) and (2), p. 3, about *concept definitions*, *concept representations* and *connections* with vectors, it seems that *vectors* had primary *geometric* nature in connection with concepts in physics.

In comparison with Gauss's study of the geometric representations of complex numbers, algebraic representations of complex numbers as ordered pairs of real numbers were studied by **Hamilton** (1805-1865). His last book, *Elements of Quaternions*, published in 1866 is considered as important for the historical development of vectors and also other concepts as the *dot product* and the *cross product* of vectors. For these reasons, I will discuss it more in details. It consists of three books. The First Book deals with the *Concept of Vector*, considered as *a directed right line*, in three-space. The Second Book introduces a *First Conception of a Quaternion*, considered as a 'quotient' of two vectors. And the Third Book treats of *Products and Powers of Vectors*, regarded as constituting a "Second Principal Form of the Conception of Quaternions in Geometry" ([Hamilton, 1866], p. vii).



ELEMENTS OF QUATERNIONS.

BOOK I.

ON VECTORS, CONSIDERED WITHOUT REFERENCE TO ANGLES, OR TO ROTATIONS.

CHAPTER I.

FUNDAMENTAL PRINCIPLES RESPECTING VECTORS.

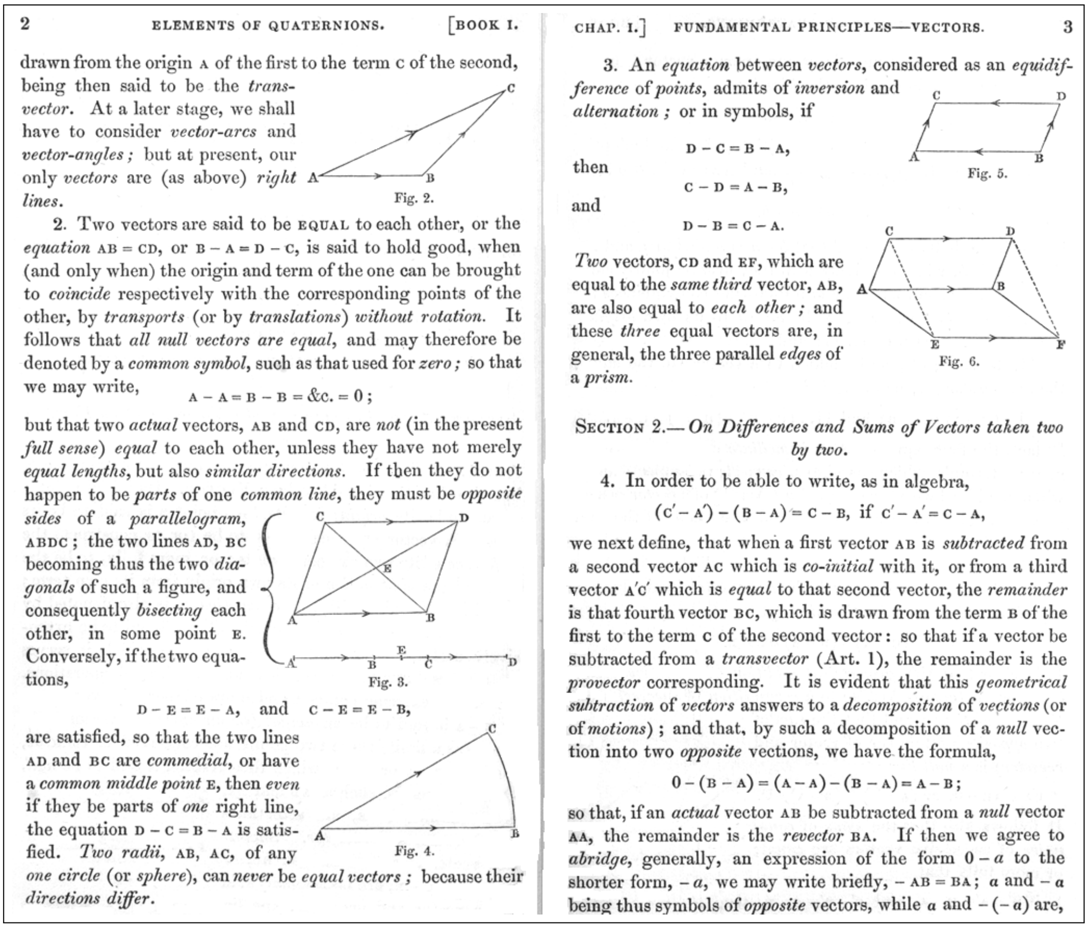
SECTION 1.— *On the Conception of a Vector; and on Equality of Vectors.*

ART. 1.—A right line AB, considered as having not only *length*, but also *direction*, is said to be a VECTOR. Its initial point A is said to be its *origin;* and its final point B is said to be its *term*. A vector AB is conceived to be (or to construct) the *difference* of its two extreme points; or, more fully, to be the result of the *subtraction* of its own origin from its own term; and, in conformity with this *conception*, it is also denoted by the *symbol* B − A: a notation which will be found to be extensively useful, on account of the analogies which it serves to express between geometrical and algebraical operations. When the extreme points A and B are *distinct*, the vector AB or B − A is said to be an *actual* (or an *effective*) vector; but when (as a limit) those two points are conceived to *coincide*, the vector AA or A − A, which then results, is said to be *null*. *Opposite* vectors, such as AB and BA, or B − A and A − B, are sometimes called vector and *revector*. *Successive* vectors, such as AB and BC, or B − A and C − B, are occasionally said to be vector and *provector:* the line AC, or C − A, which is

Figure 1.7: Basic Terms about Vectors ([Hamilton, 1866], p. vii)

In the First book, Hamilton defines a vector as "a right line AB" having both length and direction (Figure 1.7). This definition points out the *geometric* nature of the concept, which is the reason why further considerations are often accompanied with

geometric visualizations (Figures 1.7, 1.8 and 1.11). Further on, he defines basic terms about vectors, called by him as "fundamental principles respecting vectors" ([Hamilton, 1866], p. vii, Figure 1.7).
He introduces the vector concept with an emphasis that a clear distinction must be made between the terms "vector and revector", "provector", "transvector", "actual and null vector", "origin and term of a vector", "equal and unequal vectors" etc. (Figure 1.7). These Hamilton's terms refer to today's terms: "vector and opposite vector", "consecutive vector", "sum vector", "non-zero and zero vectors", "initial and terminal point of the representative of a vector" and "equal and non-equal vectors", respectively.
Hamilton's algebraic approach to vectors in close connections to their geometric illustrations can be seen in the way he defines not only basic terms about vectors but also vector operations such as addition and subtraction ([Hamilton, 1866], p. 2-3, Figure 1.8). For example, triangle and parallelogram rules for addition of two vectors together with their geometric representations, as well as an addition of three non-collinear vectors can be seen in the same Figure 1.8 ([Hamilton, 1866], p. 2-3).



2 ELEMENTS OF QUATERNIONS. [BOOK I.

drawn from the origin A of the first to the term C of the second, being then said to be the *transvector*. At a later stage, we shall have to consider *vector-arcs* and *vector-angles*; but at present, our only *vectors* are (as above) *right lines*.

Fig. 2.

2. Two vectors are said to be EQUAL to each other, or the *equation* AB = CD, or B − A = D − C, is said to hold good, when (and only when) the origin and term of the one can be brought to *coincide* respectively with the corresponding points of the other, by *transports* (or by *translations*) *without rotation*. It follows that *all null vectors are equal*, and may therefore be denoted by a *common symbol*, such as that used for *zero*; so that we may write,

$$A - A = B - B = \&c. = 0;$$

but that two *actual* vectors, AB and CD, are *not* (in the present *full sense*) *equal* to each other, unless they have not merely *equal lengths*, but also *similar directions*. If then they do not happen to be *parts* of one *common line*, they must be *opposite sides* of a *parallelogram*, ABDC; the two lines AD, BC becoming thus the two *diagonals* of such a figure, and consequently *bisecting* each other, in some point E. Conversely, if the two equations,

Fig. 3.

$$D - E = E - A, \quad \text{and} \quad C - E = E - B,$$

are satisfied, so that the two lines AD and BC are *commedial*, or have a *common middle point* E, then *even* if they be parts of *one* right line, the equation D − C = B − A is satisfied. *Two radii*, AB, AC, of any *one circle* (or *sphere*), can *never* be *equal vectors*; because their *directions differ*.

Fig. 4.

CHAP. I.] FUNDAMENTAL PRINCIPLES—VECTORS. 3

3. An *equation* between *vectors*, considered as an *equidifference* of *points*, admits of *inversion* and *alternation*; or in symbols, if

$$D - C = B - A,$$

then

$$C - D = A - B,$$

and

$$D - B = C - A.$$

Fig. 5.

Two vectors, CD and EF, which are equal to the *same third* vector, AB, are also equal to *each other*; and these *three* equal vectors are, in general, the three parallel *edges* of a *prism*.

Fig. 6.

SECTION 2.—*On Differences and Sums of Vectors taken two by two.*

4. In order to be able to write, as in algebra,

$$(C' - A') - (B - A) = C - B, \text{ if } C' - A' = C - A,$$

we next define, that when a first vector AB is *subtracted* from a second vector AC which is *co-initial* with it, or from a third vector A′C′ which is *equal* to that second vector, the *remainder* is that fourth vector BC, which is drawn from the term B of the first to the term C of the second vector: so that if a vector be subtracted from a *transvector* (Art. 1), the remainder is the *provector* corresponding. It is evident that this *geometrical subtraction of vectors* answers to a *decomposition* of *vections* (or of *motions*); and that, by such a decomposition of a *null* vection into two *opposite* vections, we have the formula,

$$0 - (B - A) = (A - A) - (B - A) = A - B;$$

so that, if an *actual* vector AB be subtracted from a *null* vector AA, the remainder is the *revector* BA. If then we agree to *abridge*, generally, an expression of the form $0 - a$ to the shorter form, $-a$, we may write briefly, $-AB = BA$; a and $-a$ being thus symbols of *opposite* vectors, while a and $-(-a)$ are,

Figure 1.8: Addition of Vectors ([Hamilton, 1866], p. 2-3)

Then he brings "the fundamental principles of vectors" into context with other geometric concepts as points and lines in a given plane (Figure 1.9). What is interesting here is the switch from synthetic to *coordinate geometry.* For example, points on a plane are represented with their three coordinates and later using these coordinates he forms a linear equation, which associates to an arithmetic - algebraic representation. Then he continues with a definition of a line through two given points on a plane using the notation of determinants[6] (Figure 1.9). Finally, he ends with an algebraic definition of a line (without a use of coordinates). This example points out Hamilton's mixed approach mainly characterized by a combination of coordinate geometry and algebra. It seems to be present in the most of his published works ([Hamilton, 1853, Hamilton, 1866]). It is also present in his work on quaternions, as is described below.

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37. If P_1 and P_2 be any two points in the given plane,

$$P_1 = (x_1, y_1, z_1), \qquad P_2 = (x_2, y_2, z_2),$$

and if t and u be any two scalar coefficients, then the following *third point*,

$$P = (tx_1 + ux_2,\ ty_1 + uy_2,\ tz_1 + uz_2),$$

is *collinear* with the two former points, or (in other words) is situated *on the right line* P_1P_2. For, if we make

$$x = tx_1 + ux_2, \qquad y = ty_1 + uy_2, \qquad z = tz_1 + uz_2,$$

and

$$\rho_1 = \frac{x_1 \alpha a + ..}{x_1 a + ..}, \qquad \rho_2 = \frac{x_2 \alpha a + ..}{x_2 a + ..}, \qquad \rho = \frac{x \alpha a + ..}{x a + ..},$$

these *vectors* of the three points P_1P_2P are connected by the *linear equation*,

$$t\,(x_1 a + ..)\,\rho_1 + u\,(x_2 a + ..)\rho_2 - (xa + ..)\,\rho = 0\,;$$

in which (comp. 23), the *sum* of the *coefficients* is *zero*. Conversely, the point P cannot be collinear with P_1, P_2, unless its co-ordinates admit of being thus expressed in terms of theirs. It follows that if a *variable point* P be obliged to *move along a given right line* P_1P_2, or if it have such a *line* (in the given plane) for its *locus*, its *co-ordinates* xyz must satisfy a *homogeneous equation of the first degree, with constant coefficients ;* which, in the known notation of determinants, may be thus written,

$$0 = \begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix};$$

or, more fully,

$$0 = x\,(y_1 z_2 - z_1 y_2) + y\,(z_1 x_2 - x_1 z_2) + z\,(x_1 y_2 - y_1 x_2)\,;$$

or briefly,

$$0 = lx + my + nz,$$

where l, m, n are *three constant scalars*, whereof the *quotients* determine the *position* of the right *line* Λ, which is thus the *locus* of the point P. It is natural to call the *equation*, which

Figure 1.9: Equation of a Line through Two Points on a Plane ([Hamilton, 1866], p. 28)

[6]This idea is used for a design of an applet - based task in Chapter 4 of the thesis.

Hamilton's goal was to generalize complex numbers in order to describe the three-dimensional space. He tried to interpret quaternions geometrically but worked mainly with algebraic tools resulting in meaningful calculations with quaternions. He considered a quaternion $Q = w+xi+yj+zk$ consisting of a scalar part $S(Q) = w$ and an imaginary part, $I(Q) = xi + yj + zk$, thus as the sum of a scalar and a vector ("Scalar plus Vector equals Quaternion", [Hamilton, 1866], p. 11 and p. 314). In his geometric interpretation, the vector part can be represented as a directed segment in space, where the three imaginary units i, j and k build an orthogonal system. Then the usual vector addition is described as vector addition in space. Multiplying only the imaginary parts of quaternions, when the scalar part disappears, Hamilton sets the basis of today's vector operations: addition (triangle and parallelogram rules), subtraction, scalar multiplication, dot product and cross product of vectors. When the product of the imaginary parts $a = xi + yj + zk$ and $a' = x'i + y'j + z'k$ of two quaternions is decomposed to its scalar and vector parts, then the obtained result represents the two products, in today's sense: the scalar part $S(aa') = xx' + yy' + zz'$ is the (negative) *dot product* and the vector part $V(aa') = i(yz' - zy') + j(zx' - xz') + k(xy' - yx')$ is the *cross product* of vectors ([Wittmann, G., 2003]; [Crowe, 2002]). Both of these Hamilton's definitions of dot product and cross product of vectors refer to today's arithmetic-algebraic definitions (with a use of components of vectors). Moreover, he interpreted the vector part as an oriented segment, perpendicular to the two given ones, which can be considered as a trial of combining his method with a synthetic geometric interpretation. This unifying perspective was due to Hamilton's algebraic introduction of quaternions with their geometric interpretation, in contrast with his previous approach about vectors (firstly geometric and then algebraic) and in contrast to Möbius, who built a purely geometric calculus ([Wittmann, G., 2003]).

Hamilton's geometric interpretation of the algebraic introduction of the dot product and the cross product of vectors can also be seen in his publication *Lectures on Quaternions* ([Hamilton, 1853], p. (47), Figure 1.10). Here, he defines the dot product of two vectors as "the product of the lengths of the two factor-lines, multiplied by the cosine of the supplement of their inclination" ([Hamilton, 1853], p. (47), Figure 1.10). The length of the cross product of vectors is defined as a "product of the same two lengths, multiplied by the sine of the same inclination" and its direction as "perpendicular to the plane of the factor-lines" pointing out the "right-handed (or the left-handed) character" of the positions of the vectors ([Hamilton, 1853], p. (47), Figure 1.10). These two synthetic geometric definitions were born on the foundations of his previous attempts with coordinates ("triplets") (Figure 1.10).

Further analysis of Hamilton's work leads to a perception of some of the defining axioms of vector spaces, although they have not been observed as axioms by himself.

we should then be able to express the desired *product of two lines in space* by a QUATERNION, of which the constituents have very *simple geometrical significations*, namely, by the following,

$$(ix + jy + kz)\,(ix' + jy' + kz') = w'' + ix'' + jy'' + kz'',$$

where

$$w'' = -xx' - yy' - zz',$$
$$x'' = yz' - zy',\; y'' = zx' - xz',\; z'' = xy' - yx';$$

so that the part w'', independent of ijk, in this expression for the product, represents the *product of the lengths of the two factor-lines, multiplied by the cosine of the supplement of their inclination* to each other; and the remaining part $ix'' + jy'' + kz''$ of the same product of the two trinomials represents a *line*, which is in *length* the *product of the same two lengths, multiplied by the sine of the same inclination*, while in *direction* it is *perpendicular to the plane of the factor-lines*, and is such that the *rotation round the multiplier-line*, from the multiplicand-line towards the product-line (or towards the *line-part* of the whole quaternion product), has the *same right-handed* (or left-handed) *character*, as the rotation round the positive semiaxis of k (or of z), from the positive semiaxis of i (or of x), towards that of j (or of y).

[49.] When the conception, above described, had been so far unfolded and fixed in my mind, I felt that the *new instrument* for applying *calculation to geometry*, for which I had so long sought, was now, at least in part, attained. And although I had left several former conjectures respecting *triplets* for many years uncommunicated, except by name, even to friends, yet I at once proceeded to lay these results respecting *quaternions* before the

Figure 1.10: Geometric Definition of the Dot and the Cross Product of Vectors ([Hamilton, 1853], p. (47))

Namely, he had written down the following properties (Table 1.2)[7].

Today's-/ Hamilton's terminology	**Hamilton's notations**
Opposite vector/ revector	$-\alpha$ for any vector α, (p. 3) $-(-\alpha) = +\alpha$ $+(-\alpha) = -(+\alpha)$
Neutral element/ null vector	$(+\alpha) + (-\alpha) = 0$, (p. 4-5)
Commutative property of vector addition	$\alpha + \beta = \beta + \alpha$, (p. 6)
Associative property of vector addition	$(\gamma + \beta) + \alpha = \gamma + (\beta + \alpha) = \gamma + \beta + \alpha$, (p. 6)
Identity (and neutral) element	$1\alpha + 0\alpha = (1 + 0)\alpha = 1\alpha = \alpha$, (p. 8)
Distributive properties, with geometric illustration (p. 8), Figure 1.11	$n\alpha \pm m\alpha = (n \pm m)\alpha$ $m(\beta \pm \alpha) = m\beta \pm m\alpha$, (p. 8)

Table 1.2: Axioms of Vector Spaces ([Hamilton, 1866], p. 3-8)

[7]Hamilton used α, β, γ to refer to vectors and m, n to refer to scalars (see also Figure 1.11), which differs from their today's usage as all of them denoting scalars.

He also introduced the unit vector ([Hamilton, 1866], p. 117-118). Most of these axiomatic properties are accompanied by geometric visualizations. For example, the geometric illustration of the distributive property of scalar multiplication over vector addition is shown on Figure 1.11.

13. It follows that *whatever two whole numbers* (positive or negative, or null) may be represented by m and n, and *what-*

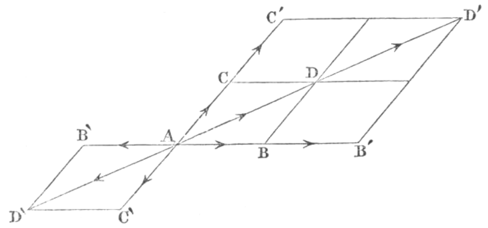

Fig. 12.

ever two vectors may be denoted by α and β, we have always, as in algebra, the formulæ,

$$n\alpha \pm m\alpha = (n \pm m)\,\alpha, \qquad n\,(m\alpha) = (nm)\,\alpha = nm\alpha,$$

and (compare Fig. 12),

$$m\,(\beta \pm \alpha) = m\beta \pm m\alpha\,;$$

Figure 1.11: Distributive Axiom of Scalar Multiplication over Vector Addition ([Hamilton, 1866], p. 8)

Although these properties were listed by Hamilton, he did not conceive them as an axiomatic system in the sense we do today. His trials to transfer the associative, distributive and non-commutative properties of multiplication of quaternions to the dot and the cross product of vectors (some of which are irrelevant, ambiguous or false) point out the need to a creation of a new algebraic system ([Crowe, 2002]). Further epistemological analysis of the historical development of vectors concepts leads to another important mathematician **Graßmann** (1809-1877). In the Foreword in the first edition of *Die Lineale Ausdehnungslehre, ein neuer Zweig der Mathematik*, 1844, (Linear Extension Theory, a New Branch of Mathematics), Graßmann says:

> The initial incentive was provided by the consideration of negatives in geometry; I was used to regarding the displacements AB and BA as opposite magnitudes. From this it follows that if A, B, C are points of a straight line, then $AB + BC = AC$ is always true, whether AB and BC are directed similarly or oppositely, that is even if C lies between A and B. In the latter case AB and BC are not interpreted merely as lengths, but rather their directions are simultaneously retained as well, according to which they are precisely oppositely oriented. Thus the distinction was drawn between the sum of lengths and the sum of such displacements in which the directions were taken into account. From this there followed the demand to establish this latter concept of a sum, not only for the

> case that the displacements were similarly or oppositely directed, but also for all other cases. This can most easily be accomplished if the law $AB + BC = AC$ is imposed even when A, B, C do not lie on a single straight line ([Graßmann, 1844], p. iv).

Graßmann's Foreword shows the birth of the concept of a vector ("a Displacement" or "Strecke") with a clear explanation of the opposite vectors and the operation vector addition for collinear and non-collinear vectors in a quite different way from Hamilton. Not only that their terminology differs (see Table 1.3), but also their approaches towards defining same concepts. Namely, Graßmann introduced vectors (the above citation) and the dot product of vectors (the citation below) in coordinate-free geometry, while Hamilton introduced concepts algebraically and interpreted them geometrically.

> Under the linear product of two displacements we understand the algebraic product of one of them and the projection of the other one over the first one, and choose the notation for linear multiplication, so that according the definition $a \cdot b = ab\cos(ab)$ ([Graßmann, 1911], p. 40).

Graßmann also defined the cross product of vectors (called by him "geometric product") in synthetic geometry, as an oriented area with particular attention to the + and - signs ([Graßmann, 1911], p. 30).

Hamilton's Terminology	**Graßmann's Terminology**	**Terminology Today**
Factor -line	Displacenent	Vector
Part "w" of a Quaternion	Linear product	Dot product of vectors
Multiplier-line	Geometric product	Cross product of vectors

Table 1.3: Hamilton's, Graßmann's and Todays Terminology

Further analysis of Grßmann' s work shows his ideas for the concepts of basis and dimension, referring to n linear independent vectors. His work stimulated **Peano** (1858-1939), who gave the first *axiomatic* definition of a real *vector space* ([Wittmann, G., 2003]). Towards the *axiomatic* definitions and approaches, the book *Vorlesungen über Zahltheorie*, 1863 from **Dirichlet** (1805-1859) speaks about the linear independent and linear dependent concepts:

> Ein System T von m Zahlen $w_1, w_2, ..., w_m$ heißt reducibel in Bezug auf einen Körper A, wenn es m Zahlen $a_1, a_2, ..., a_m$ in A giebt, die der Bedingung genügen und nicht alle verschwinden; im entgegensetzen Falle heißt das System T irreducibel nach A. Je nachdem der erste oder letztere Fall stattfindet, werden wir auch sagen, die m Zahlen $w_1, w_2, ..., w_m$ seien von einander *abhängig* oder *unabhängig* (in Bezug auf A). Ist A ein Divisor des Körpers B, so leuchtet ein, dass jedes in Bezug auf A

> reducible System auch reducibel nach B, und jedes nach B irreducible System auch irreducibel in Bezug auf A ist ([Dirichlet, 1893], p. 466).

Then Dirichlet defined the axioms for the elements of the base of the "Schaar" (space) ω with the following three axioms:

> I. Die Zahlen in ω reproduzieren sich durch Addition und Subtraction, d. h. die Summen und Differenzen von je zwei solchen Zahlen sind ebenfalls Zahlen in ω.
>
> II. Jedes Product aus einer Zahl in ω und einer Zahl in A ist eine Zahl in ω.
>
> III. Es giebt n von einander unabhängige Zahlen in ω, aber je $n + 1$ solche Zahlen sind von einander abhängig. ([Dirichlet, 1893], p. 468).

These first axiomatic definitions served as inputs in the search for more appropriate approaches in solving problems which could not be solved with the existing pure geometric or pure algebraic methods before.

In conclusion of the Sections 1.1 and 1.2 related to the introductory questions (1) and (2), p. 3, the epistemological analysis of the historical development of vector concepts shows quite opposite results from those about determinants. While determinants have arisen from arithmetic contexts in relation to solutions of systems of linear equations, first ideas for quantities different that scalars were born in contexts of physics and geometry. Studying quaternions, Hamilton introduced vectors with an arithmetic-algebraic apparatus with obligatory coordinate-geometric interpretations, whereas Graßmann used mainly synthetic geometry for vector concepts. The above analysis shows the evolution of the definitions and representations of vectors from geometric, through algebraic towards axiomatic. This is relevant for the answers to the questions in the introduction of this Chapter 1 because approaches used in the historical development of the theory of Linear algebra have their consequences in teaching and learning processes. This influence is discussed in the following section.

1.3 The Dot Product and Determinants in Upper High School

The development of the theory of vector spaces lead to their implementation in school mathematics as a part of the huge world-wide "New Math" reform in the 1950s ([Filler, 2007]). During the era of the so-called "Modern Mathematics", "what should be taught to the students of any level had to mirror the logical construction of Mathematics from that period" ([Klasa, 2010], p. 2101). The use of the axiomatic-structural approaches aimed to simplify and unify the complex theory of Linear algebra even in school but resulted in students' serious cognitive problems instead. Therefore, soon after their implementation, they started to disappear from the most national mathematics school curricula ([Dorier, 2000]; ([Klasa, 2010]). The questions that arise now are *which concepts are of an importance for the secondary level of Linear algebra and Analytic geometry in the context of axiomatic approaches* and *how could they be adequately introduced at this level of education.* The next two Subsections 1.3.1 and 1.3.2 address these two questions regarding the dot product of vectors and determinants.

1.3.1 Introduction to the Dot Product

Research studies in didactics of Linear algebra focus on the concepts of vector space, subspace, basis, dimension and linear transformations. Less frequent are studies which explore the teaching and learning of bi-linear and multi-linear forms, especially in the transition between the upper secondary- and university level of Linear algebra. Yet, an example is the book by [Tietze, Klika & Wolpers, 2000], which discusses the main mathematical ideas in Linear algebra and Analytic geometry and bridges them to school mathematics. In the overview of the "Leitideen" of Linear algebra the authors divide the content in two thematic circles. One is the theory of finite-dimensional real vector spaces and affine spaces including the corresponding linear and affine transformations and two is the theory of multi-linear forms through the finite-dimensional real vector spaces, in particular the bi-linear forms, as *the dot product*, and the multi-linear forms, as *determinants* ([Tietze, Klika & Wolpers, 2000], p. 3).

The knowledge about the concept of *linearity* preconditions the introduction to bi- and multi-linearity. It may well be studied through *vector*s. Vectors are introduced in Physics, at the lower-secondary education, before their introduction in mathematics, at the upper-secondary education. However, at the lower-secondary level, they are treated as vector entities versus scalar entities, without a precise definition in a mathematical sense. In upper-high school Linear algebra, vectors are usually introduced as classes of arrows ("Pfeilklassen", in the German literature), which are equal in length and have same direction and orientation. Besides this geometric approach, there are two other possible methods, defining vectors as ordered n-tuples, i.e. through arithmetic-algebraic representations, and defining vectors as elements of a vector space, thus axiomatic-structural approach. While the first two, the geometric and the arithmetic-algebraic representations are suitable for high school, the last one requires additional preconditions and efforts.

From a lower point of view the pedagogical importance of the concept of *dot product of vectors* comes into focus through its applications. The introduction of the dot product in high school mathematics can be motivated from applications in Physics (e.g. work equals force acting through a distance $W = \vec{F} \cdot \vec{s}$. Other motivation for the dot product of vectors can be found in *Elementary geometry*, such as the theorem: The diagonals in rhombus are orthogonal, or the Thales Theorem: An inscribed angle in a semicircle is a right angle (e.g. in the high school textbook [Bock & Walsch, 1994], p. 200). The Pythagorean Theorem is a special case of the dot product of a vector with itself and may well be used as a starting point for introducing the dot product. In contrast to these traditional ways of introduction, [Dray & Manogue, 2008] suggest a beneficial inclusion of dot product concept in *Trigonometry* courses, for example for two applications: one, the Law of Cosines can be derived immediately from the geometric definition and two, the Addition formulas (e.g. $\cos(b-a) = \cos a \cos b + \sin a \sin b$ [Dray & Manogue, 2008], p. 12) can also be derived from the dot product (as an alternative of the standard use of the unit circle definitions of trigonometric functions).

From a higher point of view, the importance of the concept of the dot product is even bigger. It is "the most important aid in metric geometry" ([Barth, E.; Krumbacher, G. & Barth, F., 1994], p. 204). Orthogonality and length (distance) are crucial notions in *Vector algebra*. A lot of problems in Analytic geometry involving length, distance and angle measures can be solved in an easier way with the help of the dot product ([Tietze, Klika & Wolpers, 2000]).

Possible approaches for introducing the dot product in upper high school are in close relation to the way vectors have previously been introduced. Namely, if vectors have been introduced as classes of arrows, which are equal in length and have same direction and orientation, then the dot product can be defined by the lengths of the vectors and the cosine of the angle between them, in other words trough the length of one of the vectors and the length of the projection of the other one. Thus, a **first approach** may be based on geometric representations. A **second approach** may base on the vectors' components, if vectors have been introduced as ordered n-tuples. In this arithmetic-algebraic approach, the dot product is defined as the sum of the products of the corresponding components of the vectors. **Approach three** may be through axiomatic properties, so as a positive symmetric bi-linear form, which is typical for a university Linear algebra course.

1.3.2 Introduction to Determinants

The teaching of multi-linearity in school may be through determinants. Yet, what is their importance?

> Der Begriff Determinante spielt in der Linearen Algebra eine zentrale Rolle. Als Hilfsmittel ist die Determinante wichtig in der Theorie der Matrizen (z. B. Inversion), der Gleichungssysteme, der linearen Abbildungen und in der Eigenwerttheorie sowie zur Einführung einer Orientierung und eines Volumenmasses in $\mathbb{R}^m$ (vgl.Abschnitt1.1.5). Es handelt sich dabei allerdings meist um Kontexte, die im wesentlichen außerhalb des Rahmens der Schulmathematik liegen. [Tietze, Klika & Wolpers, 2000],

> p. 211).[8]

In my opinion, determinants should not in any case be part of the educational processes in the upper secondary level if the intention is limited to showing the rule of Sarrus for calculating their values (for determinants in order three). The place for solving systems of linear equations and discussing solution sets is also reserved for the Gaussian algorithm and matrix transformations in the curriculum; hence, Cramer's rule based on calculations should also be avoided. However, determinants could serve as a 'rich medium' at the upper secondary level where many important concepts in Linear algebra and Analytic geometry meet. Alternative ways and new forms of visualizations could be found to treat the determinants in school as a step closer to the unified and generalized theory of Linear algebra at the university level. An environment with dynamic properties offering *multiple representations* of determinants providing different *internal* (within Linear algebra and Analytic geometry) and *external connections* (Elementary geometry, Combinatorics, Trigonometry). This could, further on, provide a base for heuristics, deep mathematical thinking, structuring, generalizing and conceptual understanding.
Looking from a higher didactical level, determinants have their values also in Calculus.

> Außerhalb der linearen Algebra ist die Determinante zum Beispiel für die Integrationstheorie für Funktionen mehrerer Variabler wichtig, weil sie eng mit dem Begriff des Volumens zusammenhängt [Jänich, 2008], p. 135).[9]

Thus, the reasons to study determinants originate in Linear algebra, Geometry and Calculus. Such different resources suggest different ways for their introduction. One typical synthetic-geometric approach at *university level* is suggested in the next paragraph.

> The *determinant* of a matrix is the (oriented) volume of the parallelepiped whose edges are its columns. If students are told this secret (which is carefully hidden in purified algebraic education), then the whole theory of determinants becomes a clear chapter of the theory of multilinear forms. If determinants are defined otherwise, then any sensible person will forever hate all determinants, Jacobians and the implicit function theorem ([Arnol'd, 1998], p. 233).

[8] The concept of a determinant plays a central role in Linear algebra. As an aid, the determinant is important in the theory of matrices (e.g. inversion), the systems of equations, and the linear transformations and in the eigenvalue theory as well as introducing orientation and a volume measure in $\mathbb{R}^m$. These are, however, mostly contexts that are essentially outside the scope of school mathematics ([Tietze, Klika & Wolpers, 2000], p. 211). Translated by the author of the thesis.

[9] Outside Linear algebra, the determinant is important for example for the integration theory for functions of several variables because it is closely related to the concept of volume [Jänich, 2008], p. 135).

This statement is worth discussing because the role of "edges" could also be played by determinants rows, besides their columns. Moreover, there is no doubt that this geometric feature of determinants is an essential part of the concept, but it should not remain unique, with complete ignorance of the algebraic features of the determinants. I personally doubt that with a pure geometric definition "the whole theory of determinants becomes a clear chapter of the theory of multi-linear forms" ([Arnol'd, 1998], p. 233). If the entries of the determinant are not real numbers, but functions or polynomials for example, then the pure geometric definition becomes senseless.
On the other hand, other researchers point out the insufficiency of pure arithmetic-algebraic models. For example, Strang (2010) starts his lecture on determinants in the second half of the university Linear algebra course with:

> Do I give you the formula for the determinant all in one gulp? I don't think so! That big formula has got too much packed in it. I would rather start with three properties of the determinant[10]. ([Strang, 2010], Transcript of the lecture 18, MIT Open Course Ware).

The formula for calculating determinants as a sum of all permutations of the set $1, 2, ..., n$ and determining the sign + or - depending on the even or odd number of interchanges of two numbers, may be a computational 'nightmare'. For large n, the sum of $n!$ terms is difficult to handle in practice and once it is done, all basic properties need to be proved. "A formula for calculation" is not the most important thing to know about a mathematical object ([Jänich, 2008], p. 136). Indeed, in the most mathematical contexts, there is no need to calculate a determinant of a matrix to two decimal places, rather a need to characterize the whole picture $\det : M_{n\times n} \to K$. This certainly does not exclude the calculation methods, but they are understood only after good knowledge for the general properties is acquired.
Thus, pure synthetic-geometric or pure arithmetic-algebraic approaches introducing determinants may be considered incomplete or inefficient. There is also another way to introduce determinants at university and it is throughout axiomatic. One can formulate simple basic properties as axioms, and derive the most important practical methods for calculation. For justification, existence and uniqueness of the determinant must be proved ([Fischer, 2011]).
Authors ([Strang, 2010]; [Fischer, 2008], 2011; [Jänich, 2008]) emphasize the advantages of the axiomatic definition of determinants in order n. From the above discussion, it is clear that the axiomatic model is definitely applicable at the uni-

[10]Strang states the following three properties in order to define the determinant:

1. $\det I = 1$.
2. Exchange rows: reverse the sign of the determinant.
3. a. $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and 3. b. $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$ ([Strang, 2010]).

versity stadium, but can it also be applied in schools? In school[11] determinants are usually defined for dimension not greater than three, but the key point is how to define them in order to serve as a base for the generalization in dimension n and avoid possible gaps in the transition.
There are actually *five* different *approaches* for introducing determinants in *upper high school* ([Tietze, Klika & Wolpers, 2000]). I label them as Approach 1 to 5 and discuss them from both historical and didactical points of views.

Approach 1: Determinants in the theory of systems of linear equations

Approach 2: As a measure for oriented areas or volumes

Approach 3: As determinant of a linear transformation

Approach 4: Determinant based on elementary row and column operations

Approach 5: Determinant characterized by a suitable axiomatic system

Approach 1 in teaching determinants is through systems of linear equations. It bases on the foundation of the work of Cramer and requires mainly skills for calculations. The historical analysis of this approach shows numerous technical difficulties for example when dealing with systems with more than three linear equations in three unknowns. The didactical analysis confirms its inappropriate usage due to the limited perspectives that it can offer such as drill and practice of computational skills. Prioritizing this approach may lead to development of procedural, rather than conceptual understanding. Thus, both historical and didactical analysis show reasons for abandoning this approach.
The **second** possible **approach** at the upper secondary level is through geometry, namely the determinant as an oriented area or volume. This approach has also roots in the historical development of the concept in the contributions of Descartes, Leibniz, Graßmann and others. Two different streams can be identified in the works of these mathematicians, one, studying algebraic concepts through coordinate geometry, as in the case of Descartes (e.g. [Descartes, 1637a] and [Descartes, 1637b]); and two, studying algebraic concepts through coordinate-free geometry, as in the case of Graßmann, (e.g. [Graßmann, 1844]). From an educational viewpoint, an isolated coordinate-free geometric approach has the disadvantage that the *internal connections* between Linear algebra and Analytic geometry, which are of great importance at this level of education, are reduced. Aiming to introduce determinants for finding areas of plane geometric figures (volumes of solids), for synthetic geometry reasons, this approach could also be considered as didactically inappropriate for the simple fact that many other effective ways exist. Students at this level of education are already familiar with formulas for areas and volume. They have studied them in elementary geometry, in lower secondary mathematics.

[11] Determinants are part of the high school Linear algebra curricula in south-east European countries and an optional topic in Berlin's mathematical gymnasiums.

Approach 3 for teaching determinants could be through linear transformations. This approach has its historical origins in the work of Frobenius, "who studied linear maps on matrix algebras preserving the determinant" ([Dolinar & Šemrl, 2002], p. 189). With this approach, determinants are defined as functions of squared matrices, which didactically means that students must have good knowledge of both matrices and functions before determinants are introduced. Thus, there exist *external connections* between concepts in Linear algebra and Calculus. Supporting this approach with geometric visualizations can be looked as an advantage.

Approach 4 for introducing determinants could be through elementary row and column operations. This approach brings elementary row operations, thus the four basic operations with numbers, in connection with vectors and areas (volumes), which means that it maintenances both *internal and external connections*. The approach may also start with the Gaussian algorithm which can be used as a strategy for converting the matrices in a triangular or diagonal form. Then, their determinants can easily be calculated. An example for introducing determinant in order two with this approach is offered in Figure 1.12.

Beispiel 12 (Herleitung der Determinantenformel und zulässige Determinantenumformungen): Der geometrische Ansatz ermöglicht mehrere Verfahren, den Wert der 2-reihigen Determinante zu bestimmen:

(a) über Zerlegungen (vgl. Bild 3.18):

$|\vec{a}\ \vec{b}| = (a_1+b_1)\cdot(a_2+b_2) - a_1a_2 - b_1b_2 - 2a_2b_1 = a_1b_2 - a_2b_1;$

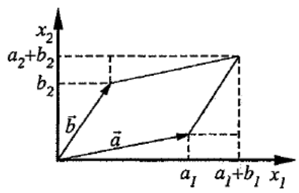

Bild 3.18

(b) über Scherungen (an einem Beispiel, vgl. Bild 3.19):

$$\begin{vmatrix}5 & 2\\ 1 & 3\end{vmatrix} = \begin{vmatrix}5 & 0\\ 1 & \frac{13}{5}\end{vmatrix} = \begin{vmatrix}5 & 0\\ 0 & \frac{13}{5}\end{vmatrix} = 13;$$

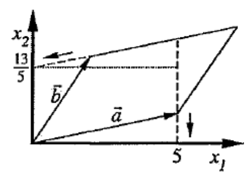

Bild 3.19

(c) über elementare Spalten- und Zeilenumformungen: Man fragt nach Spalten- und Zeilenumformungen, die den Wert der Determinante (orientiertes Flächenmaß) nicht verändern; die Begründung erfolgt abbildungsgeometrisch. Die Scherung entlang einer Parallelogrammseite bedeutet die Addition des Vielfachen einer Spalte zu einer anderen. Fragt man umgekehrt nach dem geometrischen Analogon zur Addition des Vielfachen einer Zeile zu einer anderen, so erhält man als Antwort die Scherung entlang der *x*- bzw. *y*-Achse (vgl. Schema 3.2). Mit Hilfe dieser Elementarumformungen wird die Determinante auf Diagonalform gebracht und dann berechnet. Dabei kann man an die Strategie des Gaußschen Algorithmus anknüpfen.

Figure 1.12: Determinants through Elementary Row and Column Operations ([Tietze, Klika & Wolpers, 2000], p. 212)

A clear distinction has to be made between the connections of determinants and linear transformations elaborated in Approach 4 and the determinant function representing a linear transformation itself. In Approach 4, the absolute value of a determinant equals the area of the parallelogram (volume of the parallelepiped), spanned by the vector columns and obtained as an image of the unit square (unit cube) under the linear transformation defined by the corresponding matrix. Thus, determinant preserves the orientation and area (volume).

That the determinant function represents a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$ of each of its rows (when the others are held fixed) is easily verified through the **Approach 5**[12]. This approach is discussed more in Section 1.4 because it is significant for answering the question (2) (on p. 4.) and further parts of the thesis.

Which one of these five approaches is the most suitable for upper secondary education? Can a teaching sequence be designed in such a way that it will bring together as many as possible features of the five approaches? Doesn't it seem that the axiomatic approach have the highest potential? Can an axiomatic definition provide the numerous conceptual connections and secure a natural continuation process in the transition at the same time? According to the above comparison, it seems that an appropriate model for introducing determinants at upper secondary school may be a combination of the Approaches 2, 4 and 5, i.e. an integration of geometry, algebra, and axiomatic structures. The discussion on these questions continues in section 1.4.

1.4 Axiomatic Approaches in the History and Didactics of Linear Algebra

There has been a long-time debate on the axiomatic approaches in teaching and learning mathematics. While these approaches continue to occupy the place in the textbooks and instructional materials for universities, their validity in high-school mathematics is a matter of discussion in the international community. Historically, in the era of "New Mathematics", in the 1960s, these discussions culminated with their acceptance in school. Besides the argument for the importance of the axiomatic approaches for mathematics education at university, researchers point out other possible benefits. Some of them are: "the axiomatic approach provides an important method of making the mathematics taught more elementary and the subject more restricted from the standpoint of what the student has to learn and encompass" ([Suppes, 1965], p. 2); "importance of developing intuitions for finding and giving mathematical proofs" ([Suppes, 1965], p. 3); "increasing importance of learning how to think in a mathematical fashion as the total body of mathematics itself increases so rapidly" ([Suppes, 1965], p. 4). The opponents of the axiomatic methods use the argumentation based on the difficulty of applying these methods for modelling. The debate finished in the 1990s, when extreme views contra the axiomatic approaches were founded on their inefficiency for applications in the natural sciences, particularly in physics and biology. However, from today's perspective, the axiomatic definitions remain important for university mathematics (see 1.3.1 and 1.3.2).

The abstract-structural mathematical content we expect university students to com-

[12] This feature of determinants is supported by axioms $3a$ and $3b$ in the axiomatic definition of the proposed design given in Chapter 4.

prehend should have its beginning fundamentals in school. I explain this epistemological status of the axiomatic in the following sense. I do not suggest explicit thought of the inert axiomatic formalism in school, but I expect implicit underlying knowledge of the most of the axiomatic properties of concepts by the end of upper high school (gymnasium). I can illustrate this expectation by the Peano's axiomatic. The set of natural numbers is defined with five Peano's axioms, which are never explicitly thought in primary school, yet primary school, and even kindergarten, children have an intuition about every number having a unique successor in $\mathbb{N}$, or the zero not having a predecessor in $\mathbb{N}$ (see [Gelman, 1980]; [Nordheimer, Donevska-Todorova & Henning, 2016]). Neither are lower high school students trained to acquire the Postulates of the Euclidean geometry, as an axiomatic system upon formal proofs can be derived, but are expected to be able to reason deductively in geometry. I exemplify this by the criteria for congruence of triangles: Side-Side-Side, Side-Angle-Side and Angle-Side-Angle. These examples of local axiomatics are in close connection to the Freudenthal's opinion.

> It is not because of its complexity that I blame such an axiomatic system. It is the way in which it is offered to the students. [...] Geometrical axiomatics cannot be meaningful as a teaching subject unless the student is allowed to perform these activities himself. Usually he is not allowed to do so. [...] Indeed, a student who never exercised organizing a subject matter on local levels will not succeed on the global one. [...] If a student has learned axiomatizing with easier material, he will recognize in a complicated axiomatic system the same features he knows from his former experiences, and he will be able to disentangle this system and to understand it as though he built it himself. But if axiomatizing has never been exercised, such an axiomatic system of geometry is only one more piece of indigestible mathematics ([Freudenthal, 1971], p. 426).

Freudenthal does not object axiomatic approaches in school on principal, but argues that a careful selection on the content and context which should undergo these approaches must be made ([Freudenthal, 1971], p. 432). He emphasizes that it should be done "on local levels" in order axiomatics to be 'digestible' "on the global level" ([Freudenthal, 1971], p. 426). Of course, a careful selection of particular contents at an appropriate level of students' cognition is required. Such a content may be for example the dot product, or determinants. These concepts in dimension 1, 2 and 3 may be considered crucial for the "local" effect in upper high school. The "global level", for dimension n, will later follow as a natural consequence at university.

Likewise in geometry, the introduction of concepts in Linear algebra should not ignore their existing axiomatic properties. An example of such local axiomatic for determinants is the following.

> Beispiel 12 (Charakterisierung der Determinante): Die Determinante $|\vec{a}\vec{b}|$ sei inhaltlich-konkret eingeführt als orientiertes Flächenmass des von den Spaltenvektoren $\vec{a}$ und $\vec{b}$ aufgespannten Parallelogramms. Die Addition des Vielfachen einer Spalte zu einer anderen (*) entspricht genau einer Scherung entlang einer der Parallelogrammseiten, verändert also das orientierte Flächenmass nicht. Ferner gilt $|\vec{e_1}\vec{e_2}| = 1$ (**) und schließlich $|\vec{a}\vec{b}| = -|\vec{b}\vec{a}|$ (***), weil sich durch Vertauschen der Spalten die Orientierung ändert. Man kann die Determinante nun eindeutig durch die drei Eigenschaften (*), (**) und (***) festlegen, wie man unmittelbar nachrechnet. Diese Überlegungen lassen sich auf die dreireihige Determinante übertragen. Auch für den n-dimensionalen Fall wird durch diese Eigenschaften eindeutig eine Determinante festgelegt, die aber verständlicherweise keinen konkret-inhaltlichen Bezug mehr hat ([Tietze, Klika & Wolpers, 2000], p. 112).[13]

As a part of the above suggested axiomatic for determinants, the bi-linearity property of the 2 by 2 determinant function can well be motivated by a problem-oriented access in connection with geometry (Figure 1.13)[14].

Die Bilinearität der Determinante als Funktion ihrer Spalten läßt sich ebenfalls geometrisch gewinnen (vgl. Bild 3.20). Da in dieser Sequenz zur Determinantenlehre Abbildungen eine wichtige Rolle spielen, stellt sie zugleich einen gut motivierten, problemorientierten Zugang zu einer analytischen Behandlung affiner Abbildungen dar (vgl. Schema 3.2).

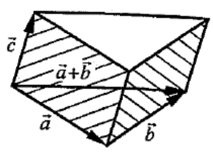

Bild 3.20

Figure 1.13: Bi-linearity of Determinants in Order Two ([Tietze, Klika & Wolpers, 2000])

Trough axiomatic systems, mathematics is seen as a deductive science, and it seems today that axiomatics are an important component of mathematical thinking which is meaningful and possible even in school [Hock, 2014].

According to the above discussion, it seems that an exclusive geometric or algebraic method is not sufficient and implementing *combined approaches* for the introduction of the concepts at the secondary level might be a better option. These new

[13] Characterization of the determinant: Be the determinant $|\vec{a}\vec{b}|$ introduced in a content-oriented concrete manner as the oriented area of the parallelogram which is spanned by the column vectors $\vec{a}$ and $\vec{b}$. The addition of a column to a multiple of another (*) corresponds exactly to shearing the parallelogram along one of its sides, thus keeping its oriented area invariant. Further $|\vec{e_1}\vec{e_2}| = 1$ (**), and finally $|\vec{a}\vec{b}| = -|\vec{b}\vec{a}|$ (***), because by inner-changing the columns the orientation changes. A determinant can now uniquely be defined by the three properties (*), (**) and (***), which can immediately be verified. These considerations can be transferred to determinant in order three. In the n-dimensional case determinants are also uniquely defined by these three properties, but of course, there exists no longer a concrete content-oriented interpretation any more ([Tietze, Klika & Wolpers, 2000], p. 112). Translation by the author of the thesis.

[14] This visualization is used for the design of an applet supporting the additive axiom $3b$ for determinants, see p. 83

approaches are supposed to contribute to the development of abstract mathematical thinking (still different than in the "New Math" sense).

Further on, there are additional facts which testify the possible benefit of combined approaches and they refer to the problem of generalization. Namely, from the algebraic perspective (related to Approach 1, p. 26), Sarus rule is not applicable for determinants in order greater than three. To calculate a determinant in order n for the aim of discussion on the existence and the solution set of a system of linear equations with Cramer's rule either the Leibniz formula with permutations or the row reduction method has to be applied. This Leibniz formula has not only technical difficulties, but also didactical ones because, it requires students' previous knowledge on permutations which are usually dealt with in stochastic. If the row reduction method is applied, then a natural continuation for solving systems of linear equations would be the Gaussian algorithm, so the Cramer's rule loses its credibility. From the geometric perspective, the generalization to n-dimensional volumes is cognitively difficult even for university students. Yet, it could intervene by "decreasing the dimension" ([Gueudet, 2004a], p. 3) to 2 or 3, particularly for upper high school students. In this sense, if an n-dimensional model is valid, then its restricted model in dimension 2 or dimension 3 is also valid and can be analysed.

I continue to discuss the problem of generalization in the next Subsection.

1.4.1 Axiomatic Definition of Determinants and Generalization to Dimension n

Definition. Let $u_1, u_2, ..., u_n$ be vectors in $\mathbb{R}^n$. The mapping $det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is called a determinant if for any $i, j = 1, 2, ..., n$ and $c \in \mathbb{R}$ the following hold:

Axiom 1. $\det \begin{pmatrix} e_1 \\ \vdots \\ e_i \\ \vdots \\ e_n \end{pmatrix} = 1$, for $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, 0, \ldots, 1)$.

Axiom 2. $\det \begin{pmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_j \\ \vdots \\ u_n \end{pmatrix} = -\det \begin{pmatrix} u_1 \\ \vdots \\ u_j \\ \vdots \\ u_i \\ \vdots \\ u_n \end{pmatrix}.$

Axiom 3a. $\det \begin{pmatrix} u_1 \\ \vdots \\ cu_i \\ \vdots \\ u_n \end{pmatrix} = c \det \begin{pmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{pmatrix}.$

Axiom 3b. $\det \begin{pmatrix} u_1 \\ \vdots \\ u_i + u_i' \\ \vdots \\ u_n \end{pmatrix} = \det \begin{pmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{pmatrix} + \det \begin{pmatrix} u_1 \\ \vdots \\ u_i' \\ \vdots \\ u_n \end{pmatrix}.$

As already mentioned in Subsection 1.3.1 (p. 26), in upper high school, determinants are introduced only in dimensions 2 and 3. A major cognitive shift is required when generalizing from dimension 3 to dimension n, which happens in the transition between the upper secondary and tertiary education level. This shift demands abstract thinking. Essential part of a successful transition is the use of algebraic representations and intuitive visual representations. For this reason the students need to master algebraic and geometric thinking during the change from dimension 2 to dimension 3 in order to manage the shift from dimension 3 to dimension n more easily. An important thing is to follow the analogy of the change between dimension 1 to dimension 2 and from dimension 2 to dimension 3 in order to become able to establish the change from dimension n to dimension $n+1$, similar to the suggested "increase of dimension" by ([Gueudet, 2004a], p. 4).

> For students to abstract a mathematical structure from a given model of that structure, the elements of that model must be conceptual entities in the student's eyes: that is to say, the student has mental procedures that can take these objects as inputs. ([Harel, 2000], p. 180).

Not only that I see this Harel's *Concreteness principle* in close connection to the Freudenthal's opinion about local axiomatic but "Harel shows in particular that geometry can provide a model helping to construct Linear algebra concepts, if the geometrical concepts are for the students mental entities on which mental operations can be performed" ([Gueudet, 2004a], p. 2). Further on, I consider the principle relevant for introducing determinants in the following sense. Objects to be taken as inputs can be determinants of order 2 and 3, which are acquired at upper secondary level, as "conceptual entities" to be generalized to dimension n at the tertiary level. The increase of the dimension has to be supported by simultaneous presence of both the algebraic and geometric modes. For an illustration of the existing analogies between 1, 2, 3 and more dimensions, here is a model which compares the geometric and the arithmetic-algebraic representations, which may support the transition from secondary to tertiary level (see Figure 1.14).

Repr.	$dim1$	$dim2$	$dim3$	$dimn$	$dimn+1$
Geom.	length	area	volume	nD volume	$n+1D$ volume
Alge.	$\lvert a \rvert$	$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$	$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$	$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$	$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} & a_{1n+1} \\ a_{21} & a_{22} & \dots & a_{2n} & a_{2n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & a_{nn+1} \\ a_{n+11} & a_{n+12} & \dots & a_{n+1n} & a_{n+1n+1} \end{vmatrix}$

Figure 1.14: Generalization of Geometric-Algebraic Representations of Determinants

In contrast to [Sierpinska, 2000], who insists on a synthetic-geometric approach when generalizing to dimension n at universities, basing her arguments on Graßmann's theory, Gueudet-Chartier states that the strong emphasis on geometry in Graßmann's book *Ausdehnungslehre* "prevented the development of the general theory" ([Gueudet-Chartier, 2004b], p. 494). Therefore, she suggest grounding Linear algebra in several domains as Geometry, Functional calculus and modern axiomatics at universities ([Gueudet-Chartier, 2004b]). In my opinion, for upper secondary education, a combination of both algebraic and coordinate-geometry aspects with additional reference to the axiomatic properties seems to be a good approach. Thus to start with an easy, non-abstract approach (not like in the "New Math"), for example with areas of parallelograms and volumes of parallelepipeds directly connected to their algebraic representations and also visualizations of the axioms for determinants, seems to have the most power in facilitating the transition towards the unified, formal nature of Linear algebra. Axioms should be an essential part of this concept; same as well as they are essential for other concepts of Linear algebra as vector spaces, dot product etc., because of their ability to integrate the features of all these concepts in a simple structure.

1.4.2 Axiomatic Approach for the Dot Product

In a similar way, the dot product should not definitely start with stating its axiomatic definition as a ready-made model, which was a typical approach in the "New Math". On the contrary, axioms should be derived in collaboration between the instructor and the students at the end of the lecture, as conclusions of students' investigations. This also meets Wittmann's idea that axiomatic systems should be a "final stage of a genetic process" ([Wittmann, E. Ch., 2009], p. 147). Thus, the introduction could start with a geometric definition for the dot product (from synthetic to coordinate) as also suggested by ([Dray & Manogue, 2008]). It could then continue with arithmetic-algebraic representations, not the other way around, as it is often done ([Dray & Manogue, 2008]), for example in textbooks ([Adam et. al., 2007], pp. 114-116; [Bock & Walsch, 1994], pp. 96-99;

[Barth, E.; Krumbacher, G. & Barth, F., 1994], pp. 211-215). This pedagogical approach can take advantage of two crucial facts. One is that geometry is an inseparable part of Linear algebra and two is that the input when finding the dot product are vectors, which are for the students mainly geometric objects (vector quantities represented by arrows in Physics, then vectors as classes of arrows in mathematics) and the output of the operation is a scalar, which can easily be calculated either by geometric or by arithmetic means (if vectors are given as n-tuples). This vector-input and scalar-output of the operation can not be treated the other way around. Axioms then emerge as a natural output of this cohesion between geometry and arithmetic. They arise as a systematic summary of what students have previously experienced and learned in Physics, lower-secondary Mathematics and upper-secondary Linear algebra.

On the base of the epistemological, historical and didactical analysis, I summarize Chapter 1 with a proposition of combined approaches for the introduction of the dot product and determinants at the secondary level Linear algebra. Such combined approaches might include the axiomatic, besides the algebraic and the geometric one, though in a significantly different way as it was done in the "New Math" era. I would rather concentrate on local levels as suggested by [Freudenthal, 1971] and [Harel, 2000], for example, by considering the axiomatic properties of the dot product and determinants. This proposition might provide insights into a possible shift in the learning trajectory of advanced concepts in Linear algebra from a higher to a lower educational level.

Chapter 2

Theoretical Framework

Besides the significance of the epistemological analysis of the historical developments of Linear algebra concepts up-to-date, other relevant theories construct the validity of the thesis. Namely, the theoretical framework is formed of mainly three theories, which are described in this chapter, *starting from the most general one towards the most specific one*. The first one, a theory for a global, long-term development of *conceptual understanding* in lower secondary, upper secondary and university education, is discussed in Section 2.1. The second relevant theory is about *concept definitions and concept images* ([Tall & Vinner, 1981]) and is discussed in Section 2.2. The third grounding theory is about *multiple modes of descriptions and thinking of Linear algebra concepts* ([Hillel, 2000]; [Sierpinska, 2000]) and is discussed in Section 2.3. The main focus of the thesis is on supporting the development of the three modes of description and thinking, which may contribute to widening students' concept images and deepen students' conceptual understanding.

2.1 Conceptual Understanding

Conceptual understanding is one of the most important constituents of mathematical proficiency ([Devlin, 2007]), but how to precisely define conceptual understanding in mathematics education and how to achieve it remains a topic in the research debate. Some researchers identify more types of students' understanding, for example, *rational* and *instrumental* types of understanding ([Skemp, 1976]), according to which, "rational understanding is knowing both what to do and why" ([Skemp, 1976], p. 2) and it is a process of "building up a conceptual structure" ([Skemp, 1976], p. 14), while instrumental understanding stands in contrast of rational understanding and it represents "rules without reason" ([Skemp, 1976], p. 2). [Nickerson, 1985] has another view on students' understanding and he identifies *results of the understanding* as: being able to see deeper characteristics of a concept, looking for specific information in a situation more quickly, being able to represent situations, and envisioning a situation using mental models, as suggested by [Barmby et al., 2007], as well. Furthermore [Nickerson, 1985] explains that understanding depends on the amount of knowledge about a subject and the conceptual contexts in which new facts are embedded ([Nickerson, 1985], pp. 235-236). Understanding can be envisioned as being "*a structure or network of mathematical ideas or representations*" and the degree of understanding depends on the number and strength of its connections ([Hiebert & Carpenter, 1992], p. 67). Understanding is the *resulting network* consisting of connections between mental representations of a mathematical concept, thus representing an action and a result of an action ([Barmby et al., 2007]). Another widely accepted categorization of understanding is to *procedural* and *conceptual* understanding ([Hiebert & Carpenter, 1992]; [Hiebert & Lefevre, 1986]). Computing skills and procedural capabilities are limited in providing sufficient conceptual understanding. In Linear algebra for example, carrying out a calculating procedure for the dot product (according a certain formula) and resulting with a scalar represents a procedural understanding, but interpretation of the obtained scalar and establishing links between it and other concepts can be considered as conceptual understanding. However, conceptual understanding is very difficult for assessment[1] ([Dubinsky & Wilson, 2013], p. 84). A similar duality has previously been offered by [Sfard, 1991], who describes the dual nature of mathematical concepts, namely *operational* and *structural*. For example, from an operational point of view, we can think of a determinant being a 'recipe' for transforming quadratic schemes of numbers, which are input ingredients, into a single number, which is an output product. From a structural point of view, a determinant is a function with well defined properties (as an axiomatic system) and can be seen as a coherent compound.

2.1.1 Guiding Features of Conceptual Understanding

There has been a lot of research from different points of views about the problem of defining conceptual understanding. In order to study upper high school students' conceptual understanding in Linear algebra and Analytic geometry in this doctoral project, I focused on five ***guiding features***. Their identification is necessary in order to locate students' possible obstacles for learning. Some of these features are similar to the categories of students' conceptual difficulties in understanding the concept of a function ([Dubinsky & Wilson, 2013], p. 85-86). Those are:

1. Do students know *what is and what is not* a concept (a vector, a dot product of vectors or a determinant? For example, do students know that a vector is a class of same directed and oriented arrows, which are equal in length, but

[1] The problem for assessment of conceptual understanding is discussed in Subsection 3.3.2.

that a single arrow is not a vector; or the dot product of vectors is a scalar, and not a vector; and a determinant is a function and not a quadratic schema of numbers?

2. Do students know that more than one definition is possible for a single concept? Do they know how are they connected? What kind of *concept definitions* and *concept images*[2] do they form? Students often think that a concept "must be defined by a single formula" ([Dubinsky & Wilson, 2013], p. 85).

3. Are students familiar with *multiple modes of description, language and thinking*[3] of Linear algebra concepts? Can they recognize and manipulate different modes and switch between them?

4. Do students understand those *properties of a concept, which construct an axiomatic definition* of the concept, the existence and uniqueness conditions in such definition?

5. Can students *connect a concept with other concepts*? For example, can they connect the dot product of two vectors and the trigonometric function cosine of an angle; or a determinant and volume of a parallelepiped?

From a historical viewpoint, features 1. and 2. are equivalent to question (1), while features 3. and 4. to question (2) in the introduction of Chapter 1, p. 3. From a viewpoint of conceptual understanding as a *structured network*, the first four of the above features refer to abilities to establish connections *within a concept* (more concept definitions of the same concept and connections between them, manipulating multiple modes of description and thinking; and applications of the concept in problem solving situations), and the last one refers to connections *with other concepts*, to integrate - within a structured network - new knowledge with the previous knowledge. Still, there is no exact borderline between these 'within' and 'with other' connections (for example, concept images are impossible without relations to other concepts), as well as there are no strict limits between the five characteristics of conceptual understanding listed above (e.g. multiple modes of description and thinking are part of a concept image). Further on, conceptual understanding is not considered as an act in a moment, rather as a long-lasting process. Deeper conceptual understanding can only be achieved through a long learning process, after a person has become fluent in procedural skills based on using the concept and following symbolical rules ([Devlin, 2007]).

[2] Concept definitions and concept images are discussed in detail in Section 2.2 of this thesis.

[3] Multiple modes of description, language and thinking are discussed in detail in Section 2.3 of this thesis.

2.2 Concept Definition and Concept Image

The meaning of *concept definition* and *concept image* in this thesis is understood as defined by [Tall & Vinner, 1981]. A *concept definition* is a "form of words used to specify the concept" and "a *personal* concept definition can differ from a *formal* concept definition, the later being a concept definition which is accepted by the mathematical community at large" ([Tall & Vinner, 1981], p. 152). A *concept image* is

> "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" ([Tall & Vinner, 1981], p. 152).

The concept image of a person consist of the set of all properties of a concept and a set of all pictures that have been associated with the concept in the person's mind ([Vinner, 1983]). [Konyalioğlu, İpek & Işik, 2003] affirm that in order to handle concepts, one needs concept images, not only concept definitions. No concept image is visible without mathematical connections. In relation to the previous discussion on conceptual understanding,

> [...] it is necessary [...] not only to have as firm a sense of the abstraction [...] but, also, a good stock of visual images for emboding them. For without latter, it is difficult to track correspondences and to check what one is doing symbolically ([Bruner & Kenney, 1965], p. 57).

Formal definitions of concepts and concept images in the field of Calculus have been discussed by many authors, as for example: the concept of *functions* ([Vinner, 1983]; [Sierpinska, 1992]; [Hoffkamp, 2010]), the concept of *integrals* ([Attorps, Björk & Radic, 2011]; [Rösken & Rolka, 2007]), the concept of *limit* ([Henning & Hoffkamp, 2013]) or the concepts of *limits and continuity* ([Tall & Vinner, 1981]). The current situation regarding discussions about students' understanding formal concept definitions and concept images in the field of Linear algebra and Analytic geometry compared to Calculus needs further wider research work covering diverse concepts at different levels of education. The scientific literature to date covers the concepts: *vector space* (e.g. [Konyalioğlu, İpek & Işik, 2003]), *subspace* (e.g. [Wawro, Sweeney & Rabin, 2011]), *linear transformations* (e.g. [Uhlig, 2003]), *binary operations* and *eigenvalues* (e.g. [Pesonen, 2003]). All these investigations in Linear algebra have been undertaken at the university level. Contributions about concepts in Linear algebra and Analytic geometry at the upper secondary education have been made by [Wittmann, G., 2003] and [Malle, 2005] regarding the concepts of *vectors*; [Tietze, Klika & Wolpers, 2000], regarding the concepts of *matrices, vector spaces, symmetric bi-linear forms, linear and affine* *transformations, determinants* and *systems of linear equations*. [Robert, 2000] also worked on different levels of conceptualizations in secondary education. [Harel, 2000] focused on students' understanding of the concept of a *vector space*.

> Students in high school deal with real numbers which are treated by them as conceptual entities. Accordingly, the symbolic representations for these objects are one-dimensional. In Linear algebra, on the other hand, new types of objects are added to the play: n-tuples, matrices, and functions as elements of a vector space. These, in contrast to real numbers, represent multidimensional quantities, [...] and they may not be conceived as conceptual entities by the students ([Harel, 2000], p. 181).

Students develop a conceptual understanding and abstractions in a concrete familiar context to them ([Harel, 2000]). Such a context serves both as an anchor to learning concept definitions and building adequate concept images ([Vinner, 1977]) and a basis for further abstractions ([Harel, 2000], p. 182).

> One of the most appealing aspects in Linear algebra, yet a serious source of difficulty for students is the "endless" number of mathematical connections one can (must) create in studying it. Relationships between systems of linear equations, matrices, linear transformations and determinants can be build in numerous ways, and problems about systems of linear equations are equivalent to problems about matrices, which, in turn are equivalent to problems of linear transformations. In this respect, Linear algebra is different from any other lower division topic in mathematics ([Harel, 1997], p. 111).

According to all sources which I consulted and considered as relevant during my study, it seems that investigations regarding students' networks of concepts such as vectors, systems of linear equations, matrices, linear transformations, determinants, lines, planes, geometric figures and solids, reserve more attention. Development of students' definitions and concept images of these concepts, influencing the conceptual understanding, with an aid of networking, need to be more explored. In this context there are three examples for concepts whose formal definitions were presented through their historical development in Chapter 1, vectors, the dot product and determinants. These examples show concept definitions and concept images of a person within conceptual networks of other mathematical concepts, which do not necessarily appear in the same way in somebody else's mental experiences with them[4]. It must be stressed that, concept definitions and concept images do not

[4] For example, regarding the concept definition in Example 1, I immediately can think of sets and two equivalence relations, parallelism and equality of length, having in mind all three properties: reflexive, symmetric and transitive. Thus a vector would be an intersection of equivalence

exclusively base on visualizations, but involve associations, relations, applications, representations, notations and ways of thinking.

Example 1. Vectors

- *Concept definition:* A vector is a class of arrows[5] which are equal in length and have same direction and orientation.[6]
- *Concept image:*
  - Vectors as arrows;
  - Vectors as n-tuples;
  - Vectors as elements of a vector space;
  - Scalar vs. Vector quantities;
  - Vectors in Physics;
  - Vector Operations;
  - Translations;
  - Matrices, determinants and multi-linear algebra, etc.

While the first three items refer to different concept definitions (geometric, arithmetic-algebraic and abstract) the rest of the items refer to applications and connections to other concepts within and out of Linear algebra. An important didactical consideration in this context is whether a student, who is offered the pointed geometric concept definition of a vector by the teacher or the text book, is able to represent it as an ordered n-tuple for example, or to recognize vector columns when learning about matrices. Such associations certainly do not happen immediately. They require longer experience with the concept.

Example 2. Dot Product of Vectors

- *Concept definition:* For two vectors $\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$ in space (analogous in plane), their dot product is the real number $a_x b_x + a_y b_y + a_z b_z$ ([Adam et. al., 2007], p 114).
- *Concept image:*

classes under each of these two relations. These associations may further lead to non-linear algebra concepts, so my personal concept images may involve knowledge from Set theory and Algebra. This certainly may not be the case with other individuals' concept images or those of the students.

[5] An arrow is an oriented segment

[6] In this example, same as in the following Example 2 and Example 3, other concept definitions are certainly possible.

  - Real number calculated trough the components of the vectors;
  - Product of the magnitudes of the vectors and the cosine of the angle between them;
  - Function satisfying three axioms;
  - Projections of vectors;
  - Angles between vectors;
  - Orthogonality;
  - Area of a certain rectangle, etc.

Similar as in Example 1, here the first three items refer to the concept definition and the rest relate to the properties and applications of the concept. Compared to the Example 1, the concept definition here is given in arithmetic-algebraic form. The pedagogical concern would be whether the student could offer a suitable visualization of such concept definition or whether the student could investigate orthogonality of two vectors applying the definition of the dot product of vectors. Analogue questions may be considered about the next Example 3.

Example 3. Determinants

- *Concept definition:* For the matrix $A = (a_{ij})_{i,j} \in K^{n\times n}$, the determinant of A is defined by

$$\det(A) = \sum_{\pi\in\sigma_n} sgn\pi \cdot \alpha_{\pi(1),1} \cdot ... \cdot \alpha_{\pi(n),n}$$

- *Concept image:*
  - Real number calculated through permutations;
  - Function from the set of square matrices to the set of real numbers satisfying certain axioms;
  - Oriented volume of parallelepipeds (oriented area of parallelograms);
  - Representations of phenomena in economy, etc.

From these examples it can be concluded that individual concept definitions and concept images substantially differ from the moment a person meets a concept for the first time till a moment after several years of personal experiences with it. This dependence on the richness of concept definitions and concept images of time can easily be illustrated for example with vectors. First, students learn about vectors in Physics, second, in high school mathematics and third, in university Linear algebra (as also illustrated in the second column of the Table 2.1, p. 45). Experiences with this concept apparently vary from practical geometric in Physics (applications for velocity, acceleration and forces), through theoretical algebraic in high school mathematics to formal abstract in university Linear algebra. These enrichments of a conceptual entity do not end with completing university studies, but continue to develop as the person invests in further, formal and post-formal education ([Tall, 2004]). Transition periods from one to a higher level of education can sometimes be fragile and deserve particular attention in this growth.

2.3 Three Modes of Description, Language and Thought in Linear Algebra

In this section, I first give an overview on how does literature describe the connections between multiple representations and conceptual understanding. Then, I specify the terminology about multiple representation and explain the meaning of multiple modes of description and thinking in Linear algebra.

Many researchers emphasize that proper combinations of representations lead to improved students' learning outcomes ([Ainsworth, et al., 1997]), translations between different representations support conceptual understanding ([Panasuk & Beyranevand, 2010]) and are important for acquiring deeper knowledge about a domain ([van der Meij & de Jong, 2006]). It is well known that quick and correct calculations or apparently fluent procedural skills are not necessarily followed by conceptual understanding. Scientific literature reports that one of the indicators of conceptual understanding is the capability for recognizing structurally the same connections posed via multiple representations ([Panasuk & Beyranevand, 2010], p. 2). Current studies identify students' difficulties in recognizing multiple representations of a single concept in Linear algebra ([Dogan-Dunlap, 2010]) and existence of limitations in students' understanding the variety representations ([Dubinsky & Wilson, 2013]). They report that even those who are able to recognize two or more representations cannot form links across them ([Dubinsky & Wilson, 2013]). They state that the algebraic representations are preferred by many students and teachers, and there is often an intention for substituting a particular representation with another even if not necessary ([Dubinsky & Wilson, 2013]). Despite these studies, students' experiences with multiple representations of concepts in Linear algebra remain an area still to be investigated ([Dogan-Dunlap, 2010]).

Let me now explain multiple representations of concepts in Linear algebra, the specific terminology, meaning and usage in this doctoral project[7]. Hillel (2000) argues that students' conceptual difficulties which are specific for Linear algebra are closely related to the existence of multiple modes of description and representation. Therefore, he defines three modes of description and language of concepts in Linear algebra: *geometric, algebraic* and *abstract* mode of description. Then, he distinguishes between a coordinate (analytic), a coordinate-free (synthetic) and a vector-as-a-

[7] The explanation offered here can also be found in ([Donevska-Todorova, 2014], pp. 305-307).

point kind of geometric mode. In the algebraic mode of description in $\mathbb{R}^n$, vectors are n-tuples of real numbers; and in the abstract mode of description, vectors are elements of a vector space V. The three "modes of description co-exist, are sometimes interchangeable, but are certainly not equivalent" ([Hillel, 2000], p. 192). Furthermore, for the purpose of establishing connections between these three modes of description, [Hillel, 2000] distinguishes between two modes of representations: *geometric-algebraic* and *algebraic-abstract* mode of representation. When changing from one mode of description into another, for example, from the geometric to the algebraic, one needs to show the compatibility of the operations (e.g. the parallelogram rule for vector addition corresponds to the component-wise vector addition). Such conversions from one into another mode are not explicitly stated in the textbooks in Linear algebra and the authors usually prefer one mode ([Pavlopoulou, 1994])[8]. Even if two modes for a single concept appear, the existing connections between them are not made explicit. Upgrading this theoretical framework, Sierpinska and co-authors([Sierpinska, 2000]; [Dreyfus, Hillel, & Sierpinska, 1998], p. 209; [Sierpinska, et al., 1997]) describe three modes of thoughts of Linear algebra concepts as follows:

- *Geometric language/ synthetic-geometric mode* of thought refers to 2- and 3-space (directed line segments, points, lines, planes, and geometric transformations).
- *Arithmetic language/ analytic-arithmetic mode* of thought refers to n-tuples, matrices, rank, solutions of systems of equations, etc.
- *Algebraic language/ analytic-structural mode* of thought refers to the general theory (vector spaces, subspaces, dimension, operators, kernels, etc.).

The three modes of thinking historically developed in a sequential manner starting with the synthetic-geometric, through the analytic-arithmetic to the analytic-structural, but without eliminations of the previous modes [Sierpinska, 2000].

Here are some examples for these modes of thinking of the concepts of vectors, the dot product and determinants.

Example 1. Vectors

- *Geometric language/ synthetic-geometric mode of thought:* Vector as a class of equal in length arrows with the same direction and orientation.

[8] Pavlopoulou uses the terminology of *registers of semiotic representations* in Linear algebra: the graphical, the table and the symbolic writing register, which are comparable to the Hillel's geometric, algebraic and abstract modes of description.

- *Arithmetic language/ analytic-arithmetic mode of thought:* Vector as an n-tuple $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$.
- *Algebraic language/ analytic-structural mode of thought:* V is a vector space (not only $\mathbb{R}^n$), every element $\vec{v} \in V$ is called a vector.

Example 2. The Dot Product of Vectors

- *Geometric language/ synthetic-geometric mode of thought:*

  $\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cos\varphi$, where φ is the angle between the vectors $\vec{u}$ and $\vec{v}$; or

  $\vec{u} \cdot \vec{v} = \pm|\vec{u}||\vec{v}_{\vec{u}}| = \pm|\vec{v}||\vec{u}_{\vec{v}}|$, where $\vec{v}_{\vec{u}}$ is a projection of the vector $\vec{v}$ over the vector $\vec{u}$ and $\vec{u}_{\vec{v}}$ is a projection of $\vec{u}$ over $\vec{v}$.
- *Arithmetic language/ analytic-arithmetic mode of thought:*

  $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + ... + u_nv_n$
- *Algebraic language/ analytic-structural mode of thought:*

  three axioms for: bi-linearity (additive and homogeneity), symmetry and positivity.

Example 3. Determinants

- *Geometric language/ synthetic-geometric mode of thought:* oriented volume (area) of parallelepipeds (parallelograms) spanned by vectors.
- *Arithmetic language/ analytic-arithmetic mode of thought:*

  $A = (a_{ij})_{i,j} \in K^{n\times n}$,

  $$\det(A) = \sum_{\pi\in\sigma_n} sgn\pi \cdot \alpha_{\pi(1),1} \cdot ... \cdot \alpha_{\pi(n),n}$$
- *Algebraic language/ analytic-structural mode of thought:* A function satisfying three axioms: multi-linearity, norm and two equal rows in a matrix, give zero value to its determinant.

Of course, other choices of the axioms satisfying the existence and uniqueness requirements are possible, but are equivalent to those offered in the above examples. A similar example about determinants is given in Subsection 3.1.3.

The first two modes of description and thought characterize the high school level of Linear algebra and Analytic geometry, while the last, analytic-structural mode of

Level of education	Vectors	Dot product of vectors	Determinants	Modes of description/ thinking in LA
Lower high school	Vector vs. scalar quantities	/	/	Geometric/ Synthetic-geometric
Upper high school	Classes of arrows	Classes of arrows	Oriented area/volume	Geometric/ Synthetic-geometric
	n-tuples (ordered pairs/triples)	Components of vectors	Sums of permutations Sarrus' rule	Algebraic/ Arithmetic
University, further education	Elements of Vector Spaces	Axioms	Axioms	Abstract/ Analytic-structural

Table 2.1: Long-Term Development of Vectors, Dot Product and Determinants

thinking characterizes the university level of Linear algebra. The long-term cognitive growth of these particular modes of description and thinking of the three concepts (in the examples 1., 2. and 3.) is given in Table 2.1.

In is important to be noticed that modes of description and thinking at a higher level in the cognitive growth do not replace previous modes ([Pegg & Tall, 2010]), rather receive a new adequate place in a wider structured puzzle. Such development and construction of a structural network of multiple modes of description and thinking (in which double oriented chords may represent recognition, translation, manipulation and utilization) is in close relation to the concept definitions and concept images (see Section 2.2). Namely, a particular concept definition uses corresponding mode of description and rich concept images are unimaginable without the equivalence among two or more concept definitions given in particular modes of description.

2.4 Potentials of Dynamic Geometry Systems in Supporting Multiple Modes of Description and Thought

During the 1990s Sierpinska and her colleagues have conducted series of projects in order to investigate how are students' difficulties in Linear algebra connected to the three modes of reasoning and the "inability to move flexibly between the three modes" ([Sierpinska, 2000], p. 209). Experimenting in the Cabri-geometry II environment with undergraduate and master students in mathematics education, she found out that they "tend to think in practical rather than theoretical way"

([Sierpinska, 2000], p. 209), meaning that they think of "mathematical concepts in terms of their prototypical examples rather than definitions" ([Sierpinska, 2000], p. 222) which has become an obstacle for the students' understanding "the notation of of linear transformations" ([Sierpinska, 2000], p. 222).
She identifies this kind of thinking as a reason why the students participating in the experiment failed to grasp "the structural theory of Linear algebra, with, among others, its axiomatic definition of linear transformations" ([Sierpinska, 2000], p. 211).
I try to synthesize the discussion.
First, Sierpinska's design used for the experiments was based on 2D synthetic-geometry. Her idea to present the structural theory of Linear algebra using exclusive coordinate-free geometry turned out to be impossible for the students to comprehend and the reason may be that "the synthetic mode belongs to the practical way of thinking, and the analytic-to the theoretical way of thinking ([Sierpinska, 2000], p. 233). It seems to me that neglecting even one of the three modes may result with obstacles for the students (see also Section 1.4). In fact, the geometric mode of thinking seems to be very challenging for the students and I argue about this in Section 3.1.
Second, during the 1990s, the Cabri-geometry did not have the algebraic features that many dynamic geometry systems (DGS) have today. For example, GeoGebra provides basic features of a CAS trying to bring together geometry, algebra and calculus ([Hohenwarter & Preiner, 2007]). The DGS now, have the characteristic for a *simultaneous dynamism* of multiple modes of description. Functional dependences between the modes which are incorporated in the DGS represent an advantage over the paper-pencil-based learning environments. Further on, DGS offer possibilities for a recognition of *invariant properties*, such as axiomatic properties of concepts in Linear algebra, which contributes in "understanding of the underlying abstract mathematical concepts" ([Leung, 2008], p. 136). In these features of the DGS, I see a possibility for supporting the learning of the axiomatic-abstract theory using both algebraic and geometric modes, without overemphasis of one over the other one.

In summary, the theoretical considerations in this Chapter 2 are the basis of this doctoral project which tries to explore students' conceptual understanding in Linear algebra trough five guiding features (p. 37). The focus is mainly on two of these features, namely, the development of concept definitions and concept images ([Tall & Vinner, 1981]) and, moreover, multiple modes of description and thinking ([Hillel, 2000]; [Sierpinska, 2000]). Implementation of these theories seems suitable for further investigations regarding students' deeper understanding of the dot product and determinants in a suggested Dynamic Geometry Environment (in Chapter 4).

Chapter 3

Research Problem, Research Questions and Methodology

I begin this chapter by a diagram (Figure 3.1) showing how chapters 1, 2 and 3 are connected with each other.

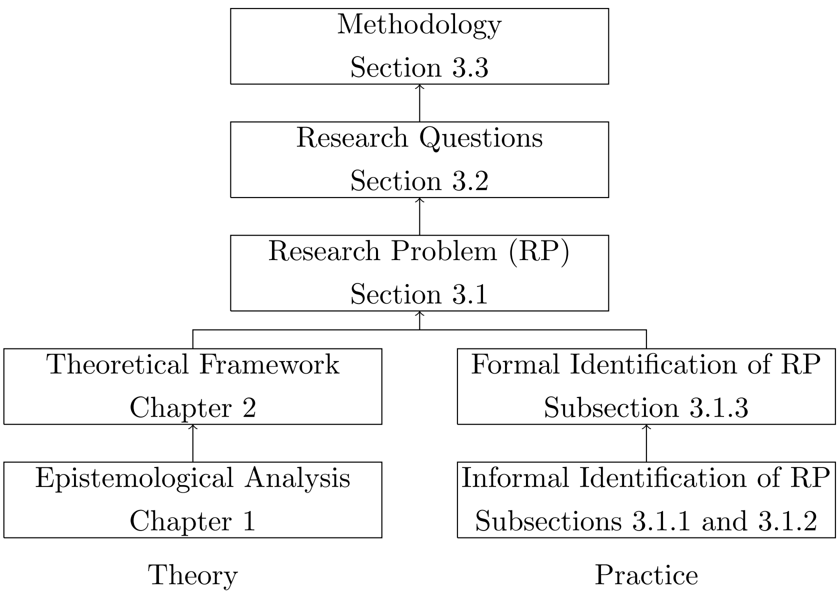

Figure 3.1: Development of Research Questions and Methodology

The diagram shows how is the *research problem* (RP) defined through theory (Chapter 1 and Chapter 2) and practice. The practical background includes informal illustration of the RP through an analysis of internet resources (Subections 3.1.1 and 3.1.2) and students work in formal university settings (Subsection 3.1.3). *Research questions* and the undertaken *methodology* aiming at answering the posed questions ([Niss, 2010]) then follow in the Sections 3.2 and 3.3.

3.1 Identification of the Research Problem

Difficulties in understanding Linear algebra concepts at the university level have been identified both in research and in practice. It is often the case that university students share their problems in understanding mathematical concepts on the internet. The following two Subections 3.1.1 and 3.1.2 identify the *research problem* through students' questions posted on on-line forums and blogs for mathematics, the first one regarding the dot product of vectors and the second one regarding determinants.

3.1.1 Students' Concept Definitions and Concept Images of the Dot Product

In this part university students' difficulties with the dot product are investigated by giving an excerpt of one mathematics forum on the internet [1]. It is shown how students personally address the problem and search for their own ways towards its solution. It is as follows.

> I am currently a high school student teacher teaching trigonometry. We are doing a unit on vectors. When inner (dot) product was taught, many questions were raised. Everyone understood that when given vector u and vector v, the dot product is $||u||$ times $||v||$ times the cosine of the angle between them, but we had a problem when we got the answer. Everyone understood that the answer was a scalar, not a vector, but **there is no graphical representation for what this scalar stands for**. I have checked *numerous sources* (every *text book* I could get my hands on, the *internet*, the *math department* at the university I am attending, and the math department where I am student teaching). I have had the students look for an answer on the internet and in the *library.* We did an *activity drawing* vectors and *comparing* the dot product with the vectors. None of us has been able to find **an understandable meaning of dot product**. We have exhausted our resources and hope you can help us. We have *done problems* involving work and the dot product, so we have *seen a real world application*, but we are still confused as to what it really is.

[1] Excerpts from the mathematics forum are given in original, as they appear on-line, thus including mistakes and non-scientific language and notations, except for the bold and italic notifications made by the author of this thesis.

> QUESTION: Exactly **what does the dot product represent? Is there a graphical explanation for the resulting scalar?** Please help us clear the confusion. Thank you. [2]

There is only one response on the posed question on the forum, namely:

> You are perhaps **thinking of dot products** the wrong way around. The dot product of the vector (x1, y1, z1) and the vector (x2, y2, z2) is written down in a few seconds:
>
> $$\text{v1.v2} = \text{x1.x2} + \text{y1.y2} + \text{z1.z2}$$
>
> Now, having this, we can find the angle between v1 and v2, since:
>
> $$\cos(theta) = \frac{v1.v2}{|v1|.|v2|}$$
>
> where $|v1| = sqrt(x1^2 + y1^2 + z1^2)$, and similarly $|v2| = sqrt(x2^2 + y2^2 + z2^2)$.
>
> By expressing v2 as a unit vector, we can also write down the component of v1 in the direction of v2. We can test for two vectors being perpendicular, since if they are perpendicular, cos(theta) = 0 and v1.v2 = 0. Since it is just as easy to work with vectors in 3 dimensions as in 2 dimensions, you will find that most 3D geometry is done using vectors, and the dot product turns up in just about every problem you can think of; for example, finding the distance of a point from a plane or from a line, or the shortest distance between two lines in space, or the equation of a plane defined by three points. Some of these can also be solved using VECTOR products, but that is a more advanced concept.
>
> In short, **we don't set out to find the dot product**. We set out to find angles between vectors, the component of a vector in some direction, the distance of a point from a line or plane, the equation of a plane, and so on and so on, and we use dot products in getting the answers to these questions. In a similar way, you don't multiply two numbers for the fun of it. You multiply numbers to answer some question which requires the technique of multiplication as an essential aid. [1]

The student teacher states the problem very precisely: "*understanding the meaning of dot product*". The first sentence in the given answer: "You are perhaps thinking of dot products the wrong way around" is worth discussing. Do "right or wrong" ways of thinking about a concept exist? Indeed, the question is exactly the essence of the problem: conceptual understanding. The arithmetic-algebraic definition offered in the second sentence of the answer, i.e. the definition in component form is

[2] [1] http://mathforum.org/library/drmath/view/52068.html. Last access on the 16.10.2015.

far away from being sufficient. However, the blame is not on the person who wrote the answer, but on the way the dot product is introduced in school, often limited to this definition and two applications: measuring lengths (dot product of a vector and itself) and measuring angles, especially orthogonality of vectors (dot product with zero value); practically two special cases of the dot product of vectors. This statement is verified in the student's claim: "we have exhausted our resources" after consulting resources in libraries, two math departments and the internet, and drawing activities, comparing and problem solving. The conclusion in the answer: "we don't set out to find the dot product" shows incompleteness of a concept definition. The importance of the dot product for its applications as calculations of angles and distances cannot be doubted. That is a fact which cannot be negated. Yet, the complete understanding of a concept starts with its concept definition, and this is often the *core of the problem*.
At this stage of the thesis, I identify the *research problem* on more than just not understanding the geometric interpretations of the dot product (the resulting scalar), but also on the:

(i) Unawareness, or incomplete awareness, of the existence of more than one *concept definition* (a geometric or an axiomatic, besides the arithmetic one) and limited *concept image*.

(ii) Exclusive focus on arithmetic-algebraic *modes of description and thought* and insufficient (if not absence of) geometric interpretation of the concept.

Similar difficulties, but now with determinants are discussed in the next Subsection 3.1.2.

3.1.2 Students' Concept Definitions and Concept Images of Determinants

This part of the thesis shows how a student asks for help not only regarding difficulties with the notation in the definition formula of determinants in order n, but also with understanding the definition itself and connecting it with other definitions. The data are gained through one mathematics portal on the internet. This example should give additional sense of the *research problem*. Neither this, nor the previous one (in Subsection 3.1.1) is meant as an empirical study, but rather as an illustration of the *research problem*.

> In my linear algebra class, we just talked about determinants. So far I've been understanding the material okay, but now I'm very confused. I get that when the determinant is zero, the matrix doesn't have an inverse. I can find the determinant of a 2×2 matrix by the formula. Our teacher showed us how to compute the determinant of an $N \times N$ matrix

> by breaking it up into the determinants of smaller matrices, and apparently there is a way by **summing over a bunch of permutations**. But the notation is really hard for me and I don't really know what's going on with them any more. Can someone help me figure out **what a determinant is, intuitively,** and **how all those definitions of it are related**? [3] [2]

What is interesting in this student's post on the blog, is that the student has a complete awareness of what (s)he knows or does not. The student clearly states that the difficulties come from the use of notation and the "summing over a bunch of permutations" for determinants in order n. It appears that the student has ambiguities in distinguishing between the formal definition (Leibniz formula through permutations), Laplace expansion of determinants in order n and how both are specified for the case of 2×2 determinants. It may be that the student thinks of Laplace expansion being a formal definition of determinants. The student is uncertain how the particular case of determinants in order two fits into the general formal definition (the problem of generalization discussed on p. 31 in Chapter 1). Even if this student is able to produce a correct formal definition, for example using permutations; the problem lies indeed in the intuitive way of thinking and understanding determinants, for which (s)he asks for help. Her/his question illustrates the need for visualizations and geometric representations, besides the knowledge about the linear (in)dependency of vectors.

In continuation, selected answers from the portal, aim to open a discussion on questions which seek deep investigations in students' current and background knowledge on the topic. The questions derive from the previously theoretically stated guiding features of conceptual understanding in Subsection 2.1.1, p. 37, but are now specified for determinants, and they are:

(a) Where all different concept definitions and concept images of determinants come from?

(b) How can all different definitions be connected?

(c) To which other concepts is the concept of determinants connected?

(d) What kind of modes of description, language and thought do students use for determinants?

[3]This question and the next quoted answers from the blog are given as they originally appear on-line, thus involving typed and notational mistakes, and informal scientific language, except for the bold and italic notification made by the author of this thesis.

[2]http://math.stackexchange.com/questions/668/whats-an-intuitive-way-to-think-about-the-determinant. Last access on the 16.10.2015.

Additionally to the theoretical background, these questions now practically help the identification of the *research problem.*
Here are some answers to the student's question posted on the same mathematical portal.
Answer 1

> Think about a scalar equation, $ax = b$ where we want to solve for x. We know we can always solve the equation if $a \neq b$, however, if $a = 0$ then the answer is "it depends". If $b \neq 0$, then we cannot solve it, however, if $b = 0$then there are many solutions (i.e. $x \in \mathbb{R}$). The key point is that the ability to solve the equation unambiguously depends on whether $a = 0$.
>
> When we consider the similar equation for matrices $Ax = b$ the question as to whether we can solve it is not so easily settled by whether $A = 0$ because A could consist of all non-zero elements and still not be solvable for $b \neq 0$. In fact, for two different vectors $y_1 \neq 0$ and $y_2 \neq 0$ we could very well have that $Ay_1 \neq 0$ and $Ay_2 = 0$
>
> If we think of y as a vector, then there are some directions in which A behaves like non-zero (this is called the *row space*) and other directions where A behaves like zero (this is called the *null space*). The bottom line is that if A behaves like zero in some directions, then the answer to the question "is $Ax = b$ generally solvable for any b?" is "it depends on b". More specifically, if b is in the *column space* of A, then there is a solution.
>
> So is there a way that we can tell whether A behaves like zero in some directions? Yes, it is the determinant! If $\det(A) \neq 0$ then $Ax = b$ always has a solution. However if,$\det(A) = 0$ then $Ax = b$ may or may not have a solution depending on b and if there is one, then there are an infinite number of solutions. [2]

Answer 2

> You could think of a determinant as a volume. Think of the columns of the matrix as vectors at the origin forming the edges of a skewed box. The determinant gives the volume of that box. For example, in 2 dimensions, the columns of the matrix are the edges of a rhombus.
>
> You can derive the algebraic properties from this geometrical interpretation. For example, if two of the columns are linearly dependent, you're box is missing a dimension and so it's been flattened to have zero volume. [2]

Answer 3

If I may, I would add to this answer (which I think is a very good one) in two minor aspects. First, a determinant also has a sign, so we want the concept of oriented volume. (This is somewhat tricky, but definitely important, so you might as well have it in mind when you're learning about "right hand rules" and such.) Second, I think better than a volume is thinking of the determinant as the multiplicative change in volume of a parallelepiped under the linear transformation. (Of course you can always take the first one to be the unit n-cube and say that you are just dividing by one. [2]

Answer 4

In addition to the answers, above, the determinant is a function from the set of square matrices into the real numbers that *preserves the operation of multiplication*: $\det(AB) = \det(A)\det(B)$ and so it carries some information about square matrices into the much more familiar set of real numbers.

Some examples:

The determinant function maps the identity matrix I to the identity element of the real numbers ($\det(I) = 1$).

Which real number does not have a multiplicative inverse? The number 0. So which square matrices do not have multiplicative inverses? Those which are mapped to 0 by the determinant function.

What is the determinant of the inverse of a matrix? The inverse of the determinant, of course. (Etc.)

This "operation preserving" property of the determinant explains some of the value of the determinant function and provides a certain level of "intuition" for me in working with matrices.[2]

Answer 5

Your trouble with determinants is pretty common. They're a hard thing to teach well, too, for two main reasons that I can see: the formulas you learn for computing them are messy and complicated, and there's no "natural" way to interpret the value of the determinant, the way it's easy to interpret the derivatives you do in calculus at first as the slope of the tangent line. It's hard to believe things like the invertibility condition you've stated when it's not even clear what the numbers mean and where they come from.

Rather than show that the many usual definitions are all the same by comparing them to each other, I'm going to state some general properties of the determinant that I claim are enough to specify uniquely what number you should get when you put in a given matrix. Then it's not too bad to check that all of the definitions for determinant that you've seen satisfy those properties I'll state.

The first thing to think about if you want an "abstract" definition of the determinant to unify all those others is that it's not an array of numbers with bars on the side. What we're really looking for is a function that takes N vectors (the N columns of the matrix) and returns a number. Let's assume we're working with real numbers for now.

Remember how those operations you mentioned change the value of the determinant? (1) Switching two rows or columns changes the sign. (2) Multiplying one row by a constant multiplies the whole determinant by that constant. (3) The general fact that number two draws from: the determinant is *linear in each row*. That is, if you think of it as a function $\det : \mathbb{R}^n \to \mathbb{R}$, then $\det(a\vec{v_1} + b\vec{w_1}, \vec{v_2}, ..., \vec{v_n}) = a\det(\vec{v_1}, \vec{v_2}, ..., \vec{v_n}) + b\det(\vec{w_1}, \vec{v_2}, ..., \vec{v_n})$, and the corresponding condition in each other slot.

I claim that these facts, together with the fact that the determinant of the identity matrix is one, is enough to define a *unique function* that takes in N vectors (each of length N) and returns a real number, the determinant of the matrix given by those vectors. I won't prove that, but I'll show you how it helps with some other interpretations of the determinant.

In particular, there's a nice geometric way to think of a determinant. Consider the unit cube in N dimensional space: the set of vectors of length N with coordinates 0 or 1 in each spot. The determinant of the linear transformation (matrix) T is the *signed volume* of the region gotten by applying T to the unit cube. (Don't worry too much if you don't know what the "signed" part means, for now). How does that follow from our abstract definition?

Well, if you apply the identity to the unit cube, you get back the unit cube. And the volume of the unit cube is 1.

If you stretch the cube by a constant factor in one direction only, the new volume is that constant. And if you stack two blocks together aligned on the same direction, their combined volume is the sum of their volumes: this all shows that the signed volume we have is linear in each coordinate when considered as a function of the input vectors.

Finally, when you switch two of the vectors that define the unit cube, you flip the orientation. (Again, this is something to come back to later if you don't know what that means).

So there are ways to think about the determinant that aren't symbol-pushing. If you've studied multi-variable calculus, you could think about,

> with this geometric definition of determinant, why determinants (the Jacobian) pop up when we change coordinates doing integration. Hint: a derivative is a linear approximations of the associated function, and consider a "differential volume element" in your starting coordinate system.
>
> It's not too much work to check that the area of the parallelogram formed by vectors (a, b) and (c, d) is $\det((a, b), (c, d))$, either: you might try that to get a sense for things. [2]

On the beginning, here is a short attempt to answer the four questions (a) to (d) stated in this section and locate the origin of the above concept definitions and concept images, and different modes of descriptions.

Answer 1 is based mainly on an algebraic concept image about determinants, because it refers to linear and matrix equations. It seems that determinants in the students' background education have been introduced by systems of linear equations, which is Approach 1, p. 26 (according to the identification in Subsection 1.3.1).

In answer 2 the geometric mode is favoured, because of the statement "You could think of a determinant as a volume". "A rhombus", further on in the text, should be a parallelogram, of course. This answer partly comes from Approach 2, p. 26, because it refers to volumes, but forgets orientation. The student has a nice attempt to "derive the algebraic properties from this geometrical interpretation", by saying "if two of the columns are linearly dependent, you're box is missing a dimension and so it's been flattened to have zero volume". This shows that the student's concept image includes some connections between Approach 2 and Approach 4.

The student who gave Answer 3 seems to have a richer geometric concept image than the previous one, because (s)he refers to determining orientation. This answer also favours the geometric mode of description and thinking and connects determinants with linear transformations (Approach 2 and Approach 3, p. 26), but ignores the algebraic aspects.

In contrast to this answer, Answer 4 seems to nave a rich algebraic concept image. The student connects determinants with matrices and mentions the important property of the determinant function, namely to preserve the operation multiplication. Student's background knowledge on the set of real numbers serves as a base for upgrading the knowledge on determinants. It helps in widening the concept image with an interesting analogy, which is established between the set of real numbers and the set of square matrices. For example, there is an analogy between the identity element 1 in $\mathbb{R}$ and the determinant of the unit matrix; or between the multiplicative inverse elements of both sets. However, this answer ignores the geometric aspects of determinants.

Answer 5 differs from the rest. First, it states that formulas for calculating determinant values are "messy and complicated", second and more important, it provides an attempt for a formal definition. This student's definition comes from Approach 5, p. 26, namely the axiomatic definition of determinants. The problem is that, "facts" (2) and (3) in the answer partly overlap, which breaks a substantial principle of axiomatic definitions. Besides this mistake, the student points out that it is "an "abstract" definition of the determinant to unify all those others" and makes a clear statement that the determinant is "not an array of numbers with bars on the side. What we're really looking for is a function that takes N vectors (the N columns of the matrix) and returns a number". This student tries to integrate the algebraic-structural mode with the geometric mode of description by naming the axioms. "In particular, there's a nice geometric way to think of a determinant. Consider the unit cube in N dimensional space: the set of vectors of length N with coordinates 0 or 1 in each spot. The determinant of the linear transformation (matrix) T is the signed volume of the region gotten by applying T to the unit cube." Furthermore, this answer shows the richest concept image, because it not only includes algebraic properties of determinants (internal connections within a concept), but also interprets them geometrically. The concept image also includes connections with many other concepts, such as matrices, vectors, oriented volume and linear transformations. Finally, it suggests why we should care about determinants, by suggesting an *external connection* to multi-variable calculus and integration.

3.1.3 Observations and Identification of the Research Problem

Observations of two lectures and three exercises sessions about determinants were undertaken at the Institute of Mathematics at the Humboldt University in Berlin. Determinants are part of the course Linear algebra and Analytic geometry II for mathematics teacher students in the first academic year of their undergraduate studies. The number of students taking this course is approximately 100. The aim of these observations was to discover which are the main *students' difficulties* in learning and understanding determinants.

The observations of the lectures and the exercises sessions are part of an *observational protocol* ([Creswell, 2013]) which is documented by keeping researcher's notes during each observation and each meeting with the lecturer and the teaching assistants. Researcher's notes include *demographic information* (time, place, date and participants), *descriptive notes* (instruction materials from the lecturer and teaching assistants and students' assignments) and *reflective notes* (reconstructions of dialogues, discussions and activities, researcher's personal thoughts, detections, ideas, proposals and impressions). The researcher neither took part in the selection of the exercises nor participated in the discussions during the lectures and the exercises sessions. In this way researchers' influence on the teaching and learning process was eliminated. These information are relevant for the discussion on the chosen methodology in Section 3.3 and on the limitations of the study in Subsection 6.2.1. All information gathered by the observational protocol represent primary material to be analysed further on.

On the beginning of the first lecture determinants in order n were defined by three axioms.

Definition: The mapping $\det : M^{n\times n} \to \mathbb{R}$ is called a determinant if the following hold:

D1 det is linear in every row.

D2 If $rgA < n$ then $\det(A) = 0$.

D3 $\det(E_n) = 1$.

In addition, examples of determinants in order two and three were given, the Cramer's rule for solving systems of linear equations with two linear equations in two unknowns was shown. A geometric representation of determinants in order two with a proof also followed. The proofs for existence and uniqueness of the determinant function satisfying the above three axioms D1, D2 and D3 were given in the second lecture. They were followed by the Laplace expansion which was pointed out as a practical way for solving some exercises. Examples of Sarrus' rule for 3×3 determinants were given at the end of the lecture.

The observations of the exercise sessions had their focus on many different aspects of determinants and their connections with other concepts, such as: matrices, invertibility of matrices, identity matrix, (non)singularity of matrices, linear (in)dependence, kernel, Gaussian algorithm, solutions of systems of linear equations and geometry. Above all, the main focus was on proper application of the definition axioms and properties of determinants, Laplace expansion (cofactor expansion), Sarrus' rule for 3 by 3 determinants and the formula for calculating determinants of triangular matrices by the product of the diagonal entries. These aspects were observed through two types of exercises: nine true/ false questions[4] and problem

[4] $\forall A, B \in \mathbb{R}^{n\times n}$ True or False:

a) $A \circ B \neq B \circ A$, but $\det(A \circ B) = \det(B \circ A)$?

b) $\det(A + B) = \det(A) + \det(B)$?

c) $\det A = 0 \Rightarrow A^{-1}$ does not exist?

d) For $A \in \mathbb{R}^{2\times 2}$, $\det A$ is the area of the spanned parallelogram by $A\vec{e_1}$ and $A\vec{e_2}$?

e) $\det A = 1 \Leftrightarrow A = E_n$?

f) $\det A \neq 0 \Leftrightarrow$ For all $b \in \mathbb{R}^n$, the system of linear equations $A\vec{x} = \vec{b}$ has exactly one solution $\vec{x}$?

g) $\det(A \circ B) \neq 0 \Leftrightarrow A$, B are regular?

h) $\det A = 0 \Leftrightarrow KernA = \vec{0}$

i) For $A \in \mathbb{R}^{3\times 3}$, $\det A$ is the volume of the spanned parallelepiped by $A\vec{e_1}$, $A\vec{e_2}$ and $A\vec{e_3}$?

solving (I come to this point later). For each of the nine questions students were given some time to think about and they were allowed to discuss in pairs. At least half of the students stated their opinion on the validity of each of the nine given statements. In some questions the majority of the students had a clear opinion and a good argumentation, as for example in the question

a) $A \circ B \neq B \circ A$, but $\det(A \circ B) = \det(B \circ A)$

in which students recognized exact use of the multiplicative property of determinants,, which was discussed during the last lecture. The most problematic (as concluded together with the teaching assistant in the meeting after the exercise session) seemed to be questions:

b) $\det(A + B) = \det(A) + \det(B)$ and

d) For $A \in \mathbb{R}^{2\times 2}$, $\det(A)$ is area of the spanned paralellogram by $A\vec{e_1}$ and $A\vec{e_2}$

Although the majority of the students stated their minds for in-correctness of the statement b), none of the students could give an own example in order to show the falsity. Neither could they explain what does it mean that a determinant is linear in every row, nor give an example to show the axiomatic property D1. Then the teaching assistant offered an example showing that the sum of two 2×2 matrices is a 2×2 matrix, while the sum of the determinants of those two matrices is a number. It aimed to make a clear distinction between addition of matrices and addition of determinants.

The second problematic question d) was about the geometric representation of determinants. On this question only three students stated their opinions: two voted for true and one for false. The rest of the students gave no answer. In order to help, the teaching assistant mentioned one determinant with a negative value, and one student concluded that the area is actually equal to the absolute value of the determinant, so that the given statement is false.
The discussion on the question:

i) For $A \in \mathbb{R}^{3\times 3}$, $detA$ is volume of the parallelepiped spanned by $A\vec{e_1}$, $A\vec{e_2}$ and $A\vec{e_3}$.

lead to a conclusion by generalizing from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{3\times 3}$ without problems.

In the second part of the lesson students had to solve problems involving determinants. I start to discuss about one of them. Students were asked to find the determinant of the matrix $I = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ as fast and rational as possible. There exist five different ways to calculate this determinant, by using: the definition axioms D1 and D3, the diagonal property for a triangular matrix, Sarrus' rule, Laplace expansion and geometric interpretation. Surprisingly, students used only three of them, non of them referring to the definition axioms and geometry. The teaching assistant interpreted the solution geometrically as a volume 8 of a parallelepiped whose sides are obtained when each side of the unit cube is stretched twice. Students' decisions about the application of the diagonal property, the Sarrus' rule and the Laplace expansion show that they used only arithmetic-algebraic approaches.

Further on, I refer to homework problems[5] , for which more adequate ways for solving are possible. During the discussions (on the meetings between the lecturer, teaching assistants and tutors) about the students' solutions of these homework problems, it was noticed that students often applied matrix elementary row operations when solving problems involving determinants and did not distinguish between determinant properties and equivalence of matrices. Changing the sign of a determinant is frequently forgotten by the undergraduate students as if they have performed matrix operations and it is one misconception of determinants ([Aygor & Ozdag, 2012]). Second remark was a predominant use of Laplace expansion in some of the homework problems. Namely students used this formula in situations when more efficient (faster and easier) ways existed. For example, in the homework problem 3a, (see footnote 5) by observing the given matrix $D = \begin{pmatrix} 2 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 2 \end{pmatrix}$ it is evident that the first and last (and also the second and the fourth) column (row) contain exactly same entries, which gives the determinant zero value. Observation of students' writings showed that only 44% of the students applied the definition axiom $D2$, p. 57, according to which $rg(D) < 5$, so $\det(D) = 0$, which very efficiently gives the correct solution. 28% of the students used Laplace expansion and the other 28% used elementary row operations to transform the given matrix in a triangular matrix and calculate the value of its determinant by multiplying the diagonal entries. There are students who applied Laplace expansion in calculating all four determinants A, B, C and D.

[5]Homework problem 3a. Compute the determinants of the following matrices

$$A = \begin{pmatrix} 3 & 4 & 6 \\ 1 & -3 & 1 \\ 9 & 0 & -13 \end{pmatrix}, B = \begin{pmatrix} 2 & -7 & 3 \\ -1 & 2 & 2 \\ 3 & 0 & -1 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 2 \end{pmatrix}. \text{ 12 pts.}$$

Similar conclusions were derived by the solutions about the determinant $C = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix}$. Just a few students applied the property: change of rows in the determinant changes its sign and the majority applied Laplace expansion.

This discourse in application of the Laplace expansion on one hand and definition axioms and properties of determinants on the other hand in practical problems may be due to different assumptions. For example, it may be due to students' predominant possession of procedural rather than conceptual knowledge ([Hiebert & Carpenter, 1992]; [Hiebert & Lefevre, 1986]) on determinants. It seems that students identify Laplace expansion as a secure way which certainly leads to a correct solution based exactly on carrying out careful number operations and technical procedures. Unlike the Laplace expansion, axioms and other determinant properties are more difficult to handle. Their implementation in problem solving situations requires abstract-structural thinking and decision making, cognitive processes which in such situations include data processing, selection of particular axiom(s) and/or property(s), justification of a certain decision and final interpretation of the result.

In the further analysis of the homework problems I mention that the inductive reasoning, when students had to verify the validity of some statements by the use definition of determinants for $n = 2$ (checked by the homework problem 1[6]), students did not have problems. This finding may be registered as students' successful manipulation of algebraic representations and mostly correct notation of determinants.

The above analysis shows that students were able to manipulate the arithmetic-algebraic representations of determinants in both situations, true/ false questions and problem solving. They successfully applied the Laplace expansion, the Sarrus' rule and the Cramer's rule (for all matrices A, B, C, D and I). It seems that the students had more problems with the geometric representations. They could not think of visualizations, thus provide geometric modes of description for determinants in order two (true/false question d) and determinants of 3×3 matrices (e.g. of the matrix I). Moreover, students rarely applied definition axioms of properties of

[6]Homework problem 1. Verify by the definition for determinant of a 2×2 matrix $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{12} - a_{21}a_{22}$:

a) Addition of a multiple of the entries in a row and the entries of another row of a 2×2 matrix A, does not change the determinant. 3 pts.

b) Change of two rows in a matrix changes the sign of its determinant. 2 pts.

c) $\det A^T = \det A$, where A^T is the transpose matrix of A. 2pts.

d) $\det(A \circ B) = \det A \cdot \det B$. 5pts.

determinants for supporting their decisions on the true/ false dilemma (question b) or in order to solve a problem (e.g. determinants of the matrices D and I).

As a consequence of this analysis, students' difficulties can be located at two main points:

(i') *the multi-linearity property of the determinant function* (students do not seem to understand what "linear in each row" means, although the axiom D1 was clearly elaborated during the lecture, students confuse the addition of determinants with the addition of matrices, students find it difficult to think of their own examples for linearity in any row),

(ii') *the geometric interpretation of determinants, thus translations between all three modes of description* (students do not understand the role of the plus or minus sign, students do not have a complete image of connections between determinants and the corresponding oriented areas of parallelograms and volumes of parallelepipeds).

These students' difficulties, particularly with the multi-linearity property and with the translation between multiple modes of description, are part of the research problem. Homogeneity and additive property of determinants are often confused with matrix operations when students multiply by a scalar or add all determinant's entries instead of entries in a single column (row). This wrong understanding is classified as one more misconception (see [Aygor & Ozdag, 2012]).

3.1.4 Research Problem

If I now compare the difficulties (i') and (ii') about determinants with those of the dot product (i) and (ii) (p. 50), I may conclude the following. The research problem is identified in the students' difficulties with the *bi-linearity* property (as exemplified by the dot product), i. e. the *multi-linearity* property (as exemplified by determinants), which include both the homogeneity and the additive properties. These are properties which construct the axiomatic definitions of the concepts. Therefore, they are in close connection with the *axiomatic-structural modes* of description and thinking. Further on, geometric visualization of the resulting scalar, regardless whether of the dot product or of the determinant, is difficult for the students. This is in relation to the *geometric mode* of description and thinking and to students' competencies for translating from one into another mode and enriching *concept images*. Therefore, a use of a DGS seems to have potentials in supporting students' geometric thinking to overcome such difficulties. In short, all of this is connected to the features of *conceptual understanding* elaborated in the theoretical framework (Chapter 2, p. 37).

3.2 Research Questions

The central research question (CRQ) and auxiliary research questions (ARQ1-5) directly address the students' learning mathematical contents in Linear algebra and Analytic geometry, precisely their conceptual understanding of vectors, of the dot product and of determinants referring to the guiding features of conceptual understanding (Subsection 2.1.1). They are as follows.

- *CRQ: How do upper high school students develope conceptual understanding of the dot product of vectors and determinants in a designed Dynamic Geometry Environment (DGE)?*

The central research question[7] does not only investigate how students learn particular concept definitions, but also what kind of concept images they form, how they connect different aspects of a concept, how they connect a concept with other concepts, how they cope with different modes of language and thinking about a concept, and how they use the concept definition and these modes of thinking for solving problems. Complexity of this research question preconditions stating auxiliary research questions (ARQ1-4) which address and contribute to answering its particular parts. They follow.

- *ARQ1: What kind of concept definitions, concept images and modes of description and thought do upper high school students evoke for vectors?*

Having in mind the importance of vector concepts for learning other concepts in Linear algebra and Analytic geometry, this ARQ1 should investigate students' previous knowledge about vectors. This knowledge is considered as necessary for further learning of the other two concepts in concern of this research study, namely the dot product and determinants.

- *ARQ2: How do upper high students develope conceptual understanding of the dot product through three modes of description and thought in a designed DGE?*

This ARQ2 points out an important operation of vectors, namely, the dot product. In contrast to operations as addition, subtraction and scalar multiplication of vectors, which students have learned in the previous years of their lower secondary education in mathematics and physics, this operation is unknown to them. For this reason, this ARQ2 investigates how students connect their previous knowledge and the new knowledge, through connections between geometric, algebraic and structural modes of description and thought.

[7]As a remark, the CRQ raises another question on how DGS have to be designed, so that they are helpful and support students' learning in a long-term and in a systematic way ([Wittmann, E. C., 2001]).

- *ARQ3: How do upper high school students learn determinants through three modes of description and thought in a designed DGE?*

This research question ARQ3 should investigate how students learn a completely new concept, determinants, and their concept definition including axiomatic properties in the designed DGE. Although determinants are considered as a difficult specific-content domain for secondary education and axiomatic approaches are considered as methodically demanding for this level of education, explorations supported by a DGE in answering such questions represent a challenge.

- *ARQ4: How do upper high school students solve problems by translating between three modes of description and thought for determinants in a designed DGE?*

After students learn concept definitions of determinants in the DGE, the next step is to investigate deeper conceptual understanding. This includes explorations on how students use concept definitions they have just learned, what kind of concept images they form on the base of different representations, how they translate between the three modes of description and thinking in order to solve problems and how they connect the new concept with other concepts in Linear algebra and Analytic geometry.

3.3 Methodology. Design-Based Research

> Some people think design means how it looks. But of course, if you dig deeper, it's really how it works (Jobs, 1996)[3].

Selection of the methodology is an overall decision from a wide range of assumptions and a creation of plans and procedures for research ([Creswell, 2013]). Complexity of the research problem, identified with university students (Section 3.1), and suggestions for preventing students' obstacles and developing conceptual understanding utilizing technology while they are still at upper high school, requires research methodology which can contribute in data analysis on *multiple level*s ([Cobb et al., 2003]). This challenge can be faced with *design-based research*[8] and for this reason it is chosen as the most suitable methodology for this study. Multiple levels of the analysis are described in Subsection 3.3.2.

[8]Appears under different labels through its historical development: design experiments ([Brown, 1992]; [Collins, 1992]; [Cobb et al., 2003]), development research ([Van der Akker, 1999]), developmental research ([Freudenthal, 1991]; [Gravemeijer, 1994], [Richey, Klein & Nelson, 2004], [Richey & Klein, 2005]), design science ([Collins, 1992]; [Wittmann, E. C., 1995]), design research ([Kelly & Lesh, 2000]; [Cobb, 2000]; [Edelson, 2002]; [Collins, Joseph, & Bielaczyc, 2004], [Kelly, 2006]) and design-based research ([The Design-Based Research Collective, 2003]; [Barab & Squire, 2004]).

Benefits of design research, in comparison with traditional research methodologies (e.g. experiments, case studies, surveys, interviews, correlation analysis), which "hardly provide prescriptions with useful solutions of a variety of design and development problems in education" ([Van der Akker, 1999], p. 2), are emphasized in literature from different aspects. Some of them are: capturing learning in rich environments ([Cobb et al., 2011]); supporting curricula designs and multilayer reform policies in education worldwide ([Van der Akker, 1999]); enabling creation of learning conditions, detected as productive in theory, but insufficiently practised or understood ([The Design-Based Research Collective, 2003], p. 5); extending current methodologies through convergence of design principals, theories and practices ([Wang & Hannafin, 2005]); and improving instructional design, development and evaluation in technology-based learning ([Van der Akker, 1999]). Design-based research is recognized as going "beyond merely designing and testing particular interventions" ([The Design-Based Research Collective, 2003], p. 6), such that, in contrast of randomized trials which may systematically fail, it embodies *theories*, *designed artefact* and *its practical implementation* in authentic classroom environments ([The Design-Based Research Collective, 2003], p. 6).

The **goal** of this design-based research is to create and practically implement an artefact for gaining conceptual knowledge, thus to provide answers to the CRQ. This design-based research was guided by *five cross-cutting features* defined in research as follows. The *first* feature of design research ([Cobb et al., 2003]; [The Design-Based Research Collective, 2003]) recognizes development of theories supporting the *process of learning* and designing *means* to support this learning. In this research, the *process of learning* includes acquiring concept definitions and widening concept images through connections between three modes of description and thinking, which contributes to developing conceptual understanding as a long lasting process. Designed *mean* supporting this learning process is an appropriate DGE. Highly interventionist and cyclic nature of the methodology, as the *second* feature of design research ([The Design-Based Research Collective, 2003]; [Cobb et al., 2003]; [Collins, 1992]), including measures of control, is described through *seven phases* of the research ([Kelly, Lesh & Baek, 2008]) elaborated in Subsection 3.3.1. The prospective and the reflective sides of the design are implemented through a hypothetical learning trajectory (HLT) (in Chapter 4) and an actual learning trajectory (ALT) (Chapter 5), respectively, as the *third* cross-cutting feature of design experiments ([Cobb et al., 2003]). The *fourth* characteristic, the iterative nature of research design ([Cobb et al., 2003]) is systematically organized through the phases, with particular attention to evidence on collected data (in Subsection 3.3.2). This study provides "detailed guidance in organizing instruction" as the last *fifth* feature ([Cobb et al., 2003], p. 10) through addressing particular parts of the designed instrument in the HLT and its implementation in the ALT.

The specification of this design-based research ([Cobb et al., 2003]) distinguishes between: conceptual understanding starting on the basis of concept definitions and concept images and continuing with usage of and translations between three modes of description and thought for solving problems, as a *target* of the investigations; DGE as an *ancillary element*; students' previous knowledge (detected with a pre-study, further given in Section 4.1), as a *background condition*; and instrumental orchestrations and instructors' roles as *accidental elements* of the design-based research.

3.3.1 The Complete Cycle of Design-Based Research

This design-based doctoral project undertakes seven phases of a complete design cycle defined by [Kelly, Lesh & Baek, 2008] which is adapted for the needs of this research study as presented on the diagram on Figure 3.2 and described in Table 3.1. Each phase is consisted of one or more processes and has certain outcome(s), which preconditions the next phase in the cycle.

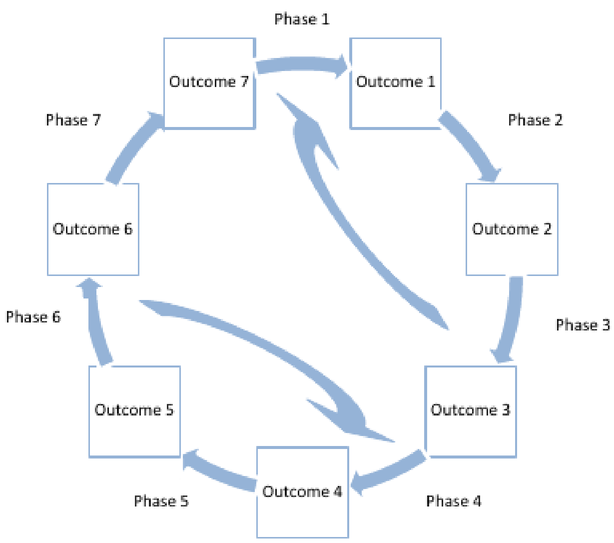

Figure 3.2: Adjusted Diagram for the Complete Cycle of Design Research in ([Kelly, Lesh & Baek, 2008], p. 32)

The Table 3.1 provides an overview on the phases of the design cycle in more detail. Phases three to six represent design experiments undertaken in school with the aid of the previously designed artefact in phase two. The last phase serves as a measure for the quality of this design and provides input for further research.

Designed instrument through all of the phases in the cycle aims to contribute in supporting students' conceptual understanding of the concepts in concern, thus to answer the main research question. This is not in any case a trivial work and therefore particular phases (or parts of the phases) directly address single ARQ, as presented in the above Table.

Phase in the Design Cycle		Outcomes of Phases in the Cycle	Answers to ARQs	Chapter
1. Overview on historical genesis, theoretical framework and identification of the Research problem		CRQ, ARQs and methodology	/	1, 2 & 3
2. Development of artefact		Designed artefact and HLT	/	4
3. Feasibility study	Design Experiments	Pre-study, pilot trial and small-scale interventions	ARQ1: Pre-study	5
4. Prototyping and trials		Iterations between testing and designs	ARQ2: Unit 1	
5. Field study		ALT	ARQ3: Unit 2	
6. Definite test		Test and assessment	ARQ4: Unit 3	
7. Dissemination and impact		On-line public release	Conclusions	6

Table 3.1: Seven Phases of the Complete Design Cycle

3.3.2 Research Methods and Data Collection

This design-based research includes a large corpus of **data sources** generated through the phases:

1. History, research, theory, text books in Phase 1 (Chapter 1 and Chapter 2)
2. On-line posts on mathematical blogs and forums; field notes from university lectures, tutorials and team-meetings in Phase 1 (Chapter 3),
3. Pre-study in upper high school in Phase 3 (Chapter 5),
4. Voice and/ or video recordings of classroom sessions in high school, copies of students' worksheets in Phases 3 to 6 (Chapter 5)
5. Copies of students' homeworks and Mathematical journals[9] in Phases 4 and 6 (Chapter 5)
6. GeoGebra Tube in Phase 7 (Chapter 6)

These sources yield to six descriptive **data sets** of a different nature, which were analyzed systematically according the following measures:

1. The first data set should provide historical background of the problem and determine the current stadium in research on the topic by an overview on literature and text books.

[9]Mathematical journals, or diaries, are more than students' collections of data. They serve for keeping records about mathematics lessons, about what students have learned and understood or not, personal opinion, thoughts and even feelings about particular matters on a daily base. Students' Mathematical journals which were used in this project can be seen in Appendix E, p. 193.

2. The second data set, verbal and written communication of university students, should identify and illustrate the research problem at the university level, thus either confirm or disprove the problems which have been previously detected in research and theory.

3. The third data set, high school students' answers to two questions in a small-scale pre-study, should detect their actual content knowledge and the preconditions for carrying out the experiments (which should answer ARQ1, p. 62).

4. The fourth data set, high school students' verbal and written works on discussing questions and tasks, should answer the research questions ARQ2, ARQ3 and ARQ4.

5. The fifth data set should provide feedback and assessment of students' works.

6. The sixth data set should provide feedback for this design and serve as a base for further design-based research (which should undergo a new complete cycle of design).

These data are collected and organized in collaboration between the researcher, a lecturer, teaching assistants and tutors in Phases 1 and 2, and also with the help of five high school teachers in Phases 3 to 6 (Chapter 5).

Multiple levels data analysis (according to [Collins, Joseph, & Bielaczyc, 2004], p. 35) in this research includes:

- *The cognitive level:* how do individuals gain conceptual knowledge by communicating concept definitions visually and verbally and translating representations between each other (three theoretical frameworks: development of conceptual understanding; concept definitions and concept images; and three modes of language and thinking about a content-specific domain: vectors, the dot product and determinants)

- *The interpersonal level:* personal interactions between the instructor and students and among students (theoretical framework: instrumental orchestration)

- *The classroom level:* active participation of all students, interactions between participants and the designed artefact in the DGE, working atmosphere, thus a collective mathematical development in classroom community (theoretical framework: instrumental orchestration)

- *The resource level:* availability of and access to the learning resources, worksheets, applets and their integration with activities (theoretical framework: instrumental orchestration)

- *The institutional or school level:* is not a part of the analysis.

A primary analysis focuses on the content-specific domain, i.e. on mathematical concepts as vectors, the dot product and determinants and how students progress in the conceptual understanding of these particular concepts; this is the first level of the multiple-levels data analysis. In the meantime, the other levels are explored to a certain extent, due to the complexity of real-life teaching-learning processes. Investigations of all characteristics of messy situations ([Collins, Joseph, & Bielaczyc, 2004], p. 20), such as social-cultural conditions or participants' emotional reactions in classrooms, can be deepen further on.

This multifaceted data base requires **mixed methods**, mainly *qualitative* (of all six data sets, from which 3, 4 and 5 complete an *empirical study*), but also including a few *quantitative* data (for the second and the last data set) ([Brown, 1992]).

In order to undertake the multiple levels data analysis I have created instruments for assessment.

Instruments for Assessment

Phase 6 in the complete cycle of design-based research (Figure 3.2 and Table 3.1, p. 65) includes assessment of students' conceptual understanding, and additional assessment of activities which support that understanding in the designed DGE ([Forster, 2006], p. 145). Development of assessment methods for evaluating learning supported by digital technologies remains an area which still requires much research ([Drijvers et al., 2010]). Though assessment of students learning is a wide research topic, especially when learning outcomes emerge in a specific technology-based environment, here is suggestion how it could be done.

Depending on the way students' understanding is defined (see Section 2.1) there exist different proposals for its assessment. National Council of Teachers of Mathematics ([NCTM, 1989], 1995, 2000) recommends assessing students understanding through *communication* as an essential part of mathematics education which helps build meaning. Communication has an important role in clarifying and developing understanding ([Pugalee, 2001]).

Assessment through *oral* and *written* communication may have various forms, but research points out the problem that often only a limited part of student's understanding is assessed ([Barmby et al., 2007]). Instructors often examine whether student's answer is the correct result of a common technical slip or already known misconception, thus award only partial credit for a subset of student's knowledge ([Sangwin et al., 2010], p. 238) without taking environmental influences into consideration. For example, when students are asked to find dot product of vectors, is students' understanding the meaning of the obtained scalar really assessed? How and why students know (or do not know) the meaning of dot product of vectors? How can technological tools facilitate understanding this meaning and its assessment? A single standardized test seems to be vague attempt for monitoring students' conceptual understanding, besides students' developments of mathematical skills, when working in a DGE. For such reasons, "a diversity of assessment tools and strategies" are recommended ([Rosenstein et al., 1996], p. 593). Namely, *alternative assessment strategies* are: authentic performance tasks, journals, portfolios, interviews, seminars and extended projects. These strategies may answer "what", "how" and "why" questions, which is usually a challenging task for traditional assessment instruments (for example written examinations or multiple choice tests). Alternative assessment has the potential to determine accomplishments of priority educational goals which rely on deep understanding and active use of knowledge in nonlinear, complex and possibly chaotic realistic contexts of learning ([Reeves & Okey, 1996]; [Herman et al., 1992]; [Young, 1995]).

Assessment of students' conceptual understanding in this undertaken design-based research represents symbiotic use of *authentic performance tasks* and *Mathematical journals*[10].

Assessment through Authentic Performance Tasks

Authentic performance tasks[11] are tasks which may have more than one solution or one solution though more ways towards it. Not only that a creation of such tasks is difficult (see Chapter 4) but also evaluation of students' performance on them represents a challenge. Designed tasks in this project follow the facets of *authenticity*, because they require high cognitive skills for detecting, explaining and verifying mathematical phenomena in a learning environment with dynamic characteristics, connect variety of learning resources as paper-pencil and technology-based tools, provoke students' spontaneous reactions, motivate students, stimulate positive working atmosphere. During undergoing the tasks, all participants in the teaching/ learning process *perform* ("do" in the E-I-S model or act in the proceptual world, [Tall, 2003]) according to Variational Dragging Schemes (p. 82 and p. 91) and described instrumental orchestrations (Section 5.5). For these reasons, it is legitimate to refer to the proposed **Discussing questions** and **Tasks** in Sections 4.2, 4.3 and 4.4 of this thesis as authentic performance tasks.

Authentic performance tasks in this technology based learning environment both guide instruction (through HLT) and serve assessment ([Kumar, 1993]). The suggested applets do not contain assessment measures[12] directly embedded within it,

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[10]Mathematical journals (Appendix E) were previously mentioned as a fifth data source for the multiple level data analysis, p. 66.

[11]In this part of the thesis I refer to **Discussing questions** and **Tasks** given in the HLT in Chapter 4 as authentic performance tasks and explain why such a reference is possible.

[12]Displaced values of areas of geometric figures on each of the applets may be considered as check control mechanisms on whether the value of dot product of vectors or determinants is the desired one. Such feedback "help students refine their thinking" permanently ([Olive et al., 2009], p. 158). However, the applets do not offer automatically generated narrative advice for further work or other guidance, thus do not intend to serve as intelligent tutors.

which may possible serve instructors' needs for assessment information. Thus, technology does not perform the assessment by itself; it is rather the whole designed artefact which supports assessment as an integral part of instruction with primary goal students' conceptual understanding.
In order the authentic performance tasks to serve the need of assessment, students' answers and solutions are categorized as: *appropriate solution, good attempt, vague attempt* and *incorrect solution* (in the Pre-study and throughout the teaching units in Sections 5.1 to 5.4). Such categorization of the answers clearly differentiates degrees of students' achievements.

Assessment through Mathematical Journals

Mathematical journal[13] (Appendix E) is an educational tool, beneficial for *students* who write in order to learn, *instructors* who read in order to receive a wealth of information and for creation of a *student-teacher dialogue* ([Borasi & Rose, 1989]). Potential benefits for students include: increased knowledge of mathematical content, improvement in learning and problem-solving skills ([Borasi & Rose, 1989]) and also development of procedural knowledge and conceptual understanding ([Jurdak & Abu Zein, 1998]). Journals enhance students' communication of mathematical thinking ([Kostos & Shin, 2010]). They are "self-directed type" of assessment ([Lanigan, 2006], p. 38) and students are usually not awarded a grade. From the instructors' perspective, journals provide opportunities for "better evaluation and re-mediation of individual students", "feedback on the course" and "long-term instructional improvements" ([Borasi & Rose, 1989], p. 352).
Mathematical journals in this study serve the need for self-assessment of students' learning outcomes and additionally, simultaneous evaluation of the design.
The journals were distributed on a regular basis, after every lecture during the experiments but were not obligatory for students. They were structured and consisted of five items. Items 1 to 4 serve to provide feedback for the *cognitive* and *interpersonal* levels of the multiple levels data analysis. The last entry 5 (my personal opinion: overview on today's lesson, teaching methods, examples, tasks, homework problems, applications etc.) aims to provide students' feedback on the *personal*, *classroom* and *resource* level of the multiple levels data analysis.
An example for *possible* answers (hypothetical, but not suggested to the students) on the first four entries (which are similar as in ([Russek, 1998]) in the journal would be:

1. Today's lesson was: <u>Dot product of vectors</u>
2. I learned: <u>how to calculate dot product of vectors</u>
3. I understand: <u>the meaning of the obtained scalar</u>

[13] Additional information for writing students journals in disciplines other than mathematics can be found in ([Thorpe, 2004]; [Bain et al., 1999]; [Spalding et al., 2002]).

4. I do not understand: symmetric axiom of dot product of vectors

The contributions of the journals as a technique for alternative assessment in a technology rich mathematics classroom have seldom been in the focus of the literature. This study may offer some insights for further development of theories in this sense.

Chapter 4

Design of a Teaching/ Learning Sequence and a Hypothetical Learning Trajectory in a DGE (Artefact and Instrument)

The role of technology in teaching and learning Linear algebra was addressed in Section 2.4. This Chapter 4 offers insights into the dynamic geometry environment (DGE) which was designed for teaching and learning the concepts of the determinant and the dot product. "How do I organize instruction so that students develop that conception as fully as possible" is a question tackled by [Martin et al., 2010], p. 2091. A *hypothetical teaching/learning sequence* (a complete scenario consisting of definitions supported with applets, examples, questions, tasks and guiding instructions) in a DGE to be used to promote students' learning of Linear algebra at the upper secondary education, is the artefact (product) of this design. This artefact is used as an input and is further on enriched with several schemes and techniques. All these ingredients form the *instrument* [1], which is presented in the next sections of this Chapter 4. The undertaken *instrumental genesis* guided by *instrumental orchestration* towards accomplishing conceptual understanding ([Drijvers et al., 2010]), in a form of an *actual teaching/learning sequence*, is analyzed in Chapter 5.

[1] Instrument = Artefact + Schemes and Techniques, for a given type of task, by ([Drijvers et al., 2010], p. 108; [Artigue, 2002]).

4.1 Preliminary Study. Investigation of Concept Definitions, Concept Images and Modes of Description of Vectors (ARQ1)

The Preliminary study (see Appendix C, p. 187) is part of the Feasibility study undertaken in Phase 3 of the Complete Cycle of this Design-based research (Figure 3.2, p. 65 and Table 3.1, p. 66). The hypothetical teaching/ learning sequence starts with this small-scale preliminary study based on two questions (see bellow), which aim to answer the ARQ1 (Section 3.2, p. 62). The study bases on paper-pencil activities which serve as input investigations about students' concept definitions, concept images and modes of description with relation to the previously given Example 1. Vectors (vectors as classes of arrows, vectors as n-tuples and vectors as elements of a vector space, p. 40, in Section 2.2) and Example 1. Vectors (geometric, arithmetic-algebraic and analytic-structural modes of language and thought, p. 43 in Section 2.3).

Question 1. What is a vector?
Alternative question 1. How would you explain to one of your classmates what a vector is?

Question 2. What is a linear combination of vectors?
Alternative question 2. How would you explain to one of your classmates what a linear combination of vectors is?

The first question aims to investigate students' current knowledge about vectors, in particular, students' concept definitions and concept images of vectors and the second one to discover students' knowledge about linear combinations of vectors, which is necessary for the learning of the dot product and determinants. Although this pre-study is a small-scale one, answers to both questions may offer sufficient data for analysing how students describe vectors, for example through different modes of description and thinking. Further on, information about students' understanding of the concept of *linearity*, as a prerequisite for the learning of *bi-linearity* and *multi-linearity*, can be used for investigations of the dot product and determinants. The findings may then be compared with those relating the students' difficulties (i) and (ii) i.e. (i') and (ii') discussed in Subsections 3.1.1 and 3.1.3.

4.2 Suggested Approach for the Dot Product of Vectors (ARQ2)

This part of the hypothetical learning trajectory (HLT) considers one more important operation with vectors, namely dot product of vectors. It aims to contribute to widening students' concept definitions and concept images of dot product of vectors, elaborated before in Section 2.2. This part of the HLT is consisted of a few steps, first, investigations on how has the concept previously been introduced to students, second, explorations whether students can connect their previous knowledge (isolated arithmetic-algebraic and geometric definitions) with the 'new' applet-based combined geometric-algebraic approach supporting axiomatic properties offered in the DGE. The suggested approach bases on using projections of vectors and it also strengthens connections between vectors, elementary geometry (areas of plane geometric figures, namely rectangles and squares) and trigonometry. It primary uses the definition: $\vec{u} \cdot \vec{v} = |\vec{u}| \underbrace{|\vec{v}| \cos \varphi} = |\vec{u}| (\pm |\vec{v}_{\vec{u}}|)$, supported with visual dynamic characteristics of **Applet 1. Dot Product**[2] (Figure 4.1).

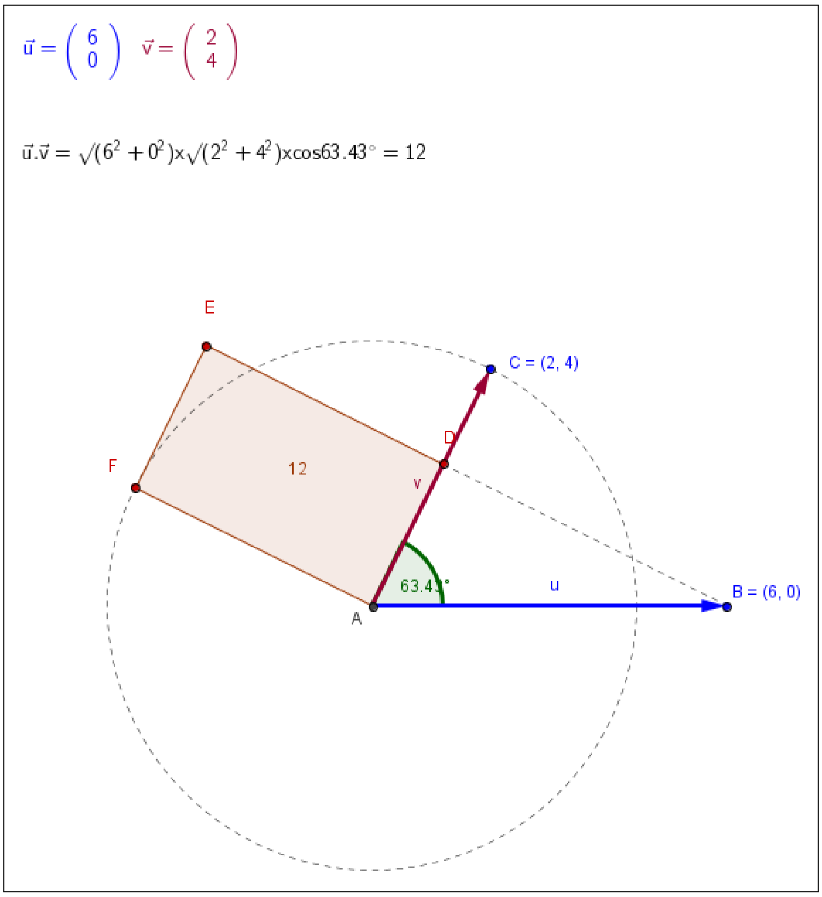


Figure 4.1: Applet 1. Dot Product

Which are possible benefits of using this applet? Fist, I give an overview of some special cases of the dot product which can be explored by the applet, and then, continue with explaining its contributions in supporting the properties of the dot product, those which construct its axiomatic definition.

[2]Snapshots of this dynamic Applet can be seen in Appendix A.

I start with a geometric definition of the dot product, which appears on the Applet 1 (Figure 4.2).

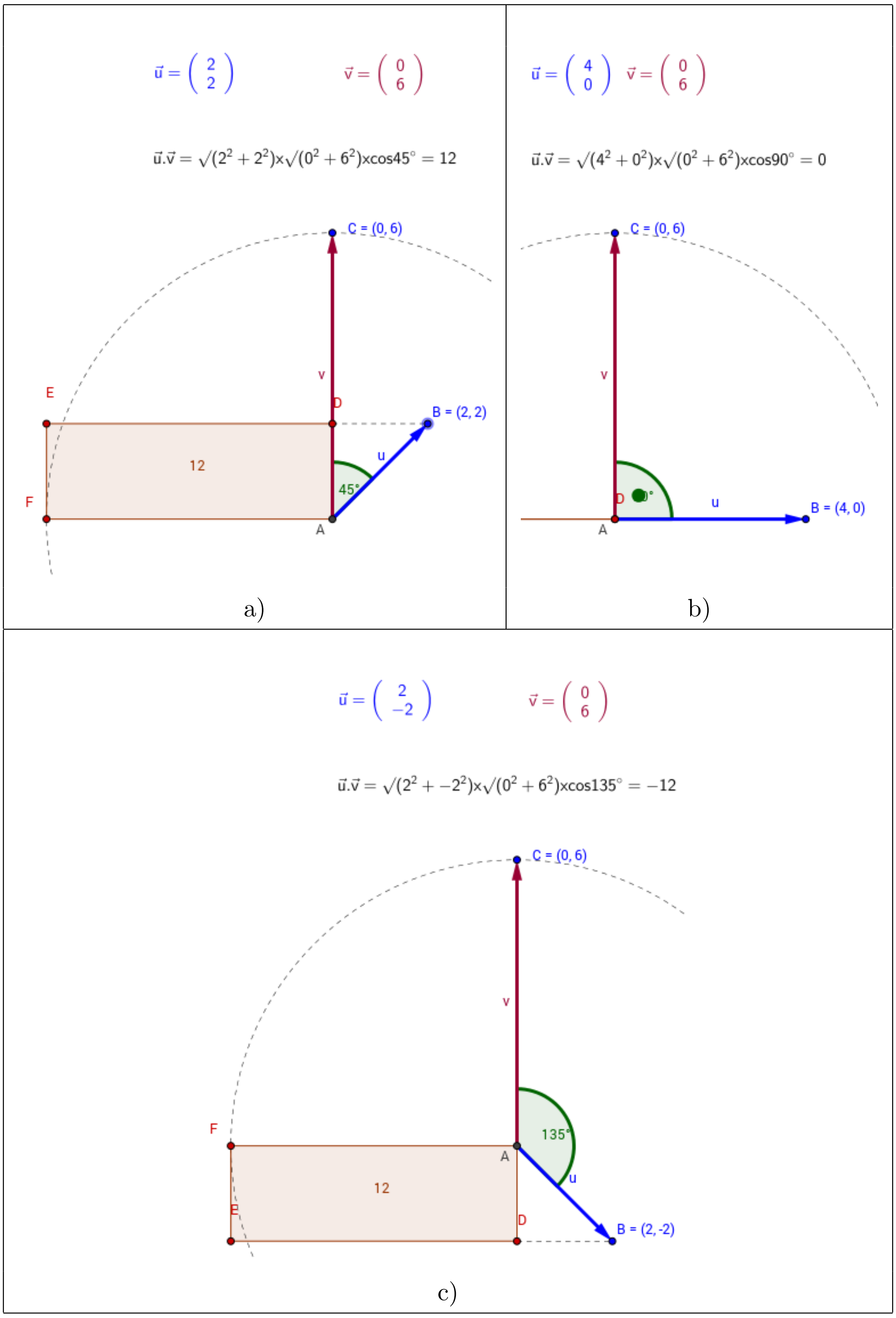


Figure 4.2: Positions of Applet 1 Showing: a) Positive, b) Zero and c) Negative Values of the Dot Product

Different positions of the Applet 1 (Figure 4.2), representing special cases in both arithmetic-algebraic and geometric modes of description of the dot product, when the angle between the vectors is acute, right and obtuse, respectively are:

1. $\vec{u} \cdot \vec{v} = |\vec{u}\,|\,|\vec{v}| \cos\varphi > 0$, $0 < \varphi < 90^\circ$ (Figure 4.2.a)

   Interpretation with this approach: the dot product is positive when one of the given vectors, for example $\vec{u}$ and the projected vector $\vec{v}$ over $\vec{u}$ (or $\vec{v}$ and the projected vector $\vec{u}$ over $\vec{v}$, as on the Figure 4.2a), have the same direction and orientation.

2. $\vec{u} \cdot \vec{v} = |\vec{u}\,|\,|\vec{v}| \cos\varphi = 0$, $\varphi = 90^\circ$ (Figure 4.2.b)

   Interpretation with this approach: the dot product is equal to zero because the magnitude of the projected vector is 0.

3. $\vec{u} \cdot \vec{v} = |\vec{u}\,|\,|\vec{v}| \cos\varphi < 0$, $90 < \varphi < 180^\circ$ (Figure 4.2.c)

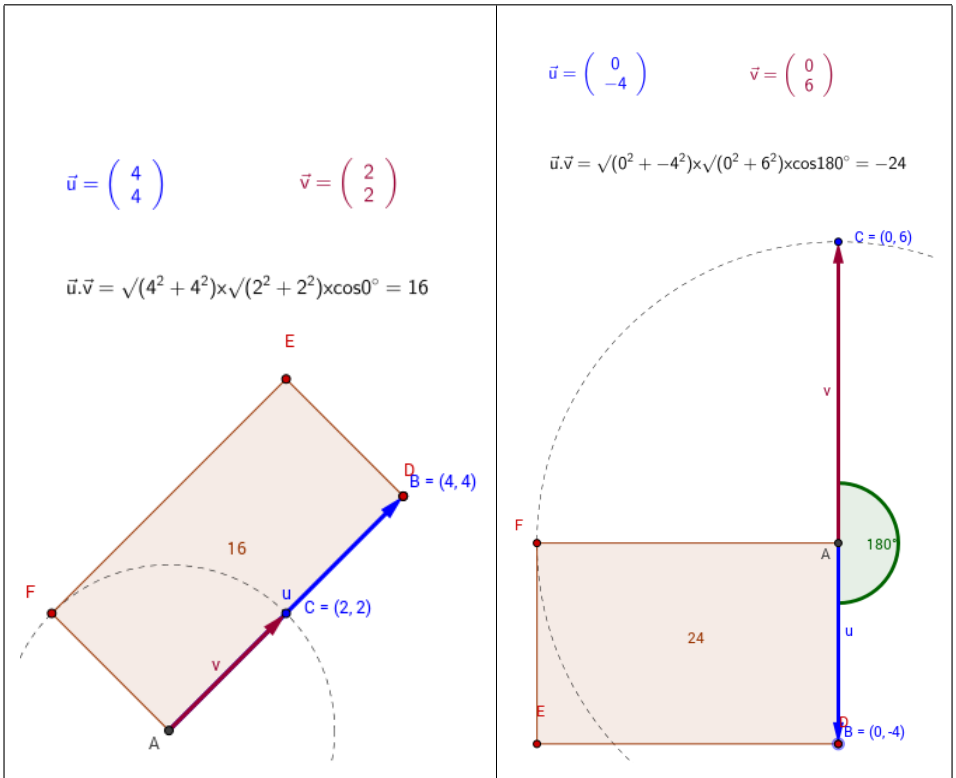
   Interpretation with this approach: the dot product is negative when one of the given vectors for example $\vec{u}$ and the projected vector $\vec{v}$ over $\vec{u}$ (or $\vec{v}$ and the projected vector $\vec{u}$ over $\vec{v}$, as on the Figure 4.2.c), have the same direction, but opposite orientations.

Further on, the applet may also contribute in reflecting on the students' knowledge in trigonometry. Namely, by setting it in positions as those shown on the Figure 4.2.a) and c), students recall their knowledge about the cosines of the angles 45° and 135° which have same values with opposite signs. In other words, they may repeat that the equality $\cos(180^\circ - \varphi) = -\cos\varphi$ holds. They may also recall that the cosine is an even function, i.e. $\cos(-\varphi) = \cos\varphi$. Simultaneously, these positions, Figure 4.2.a) and c), show same absolute value of the dot product and equal oriented areas. Likewise, similar properties of the trigonometric function cosine[3], can be repeated by setting the applet showing different angles, e.g. 225° or 315° or arbitrary angels.

Moreover, setting the applet in specific positions may prevent an occurrence of some misconceptions about the dot product. Namely, a possible misunderstanding with the special case when the angle is 0° the dot product is not 0 in non-trivial cases (because $\cos 0^\circ = 1$), can be prevented by a demonstration with the Applet 1, Figure 4.3.a).

One more special case, when the angle between the vectors is equal to 180°, is shown in Figure 4.3.b).

[3] $\cos 315^\circ = \cos(270^\circ + 45^\circ) = \cos(360^\circ - 45^\circ) = \cos 45^\circ = \sin 45^\circ$, or in general $\cos(270^\circ + \varphi) = \sin\varphi$, or $\cos(360^\circ - \varphi) = \cos\varphi$.



a) $\varphi = 0°$, positive dot product b) $\varphi = 180°$, negative dot product

Figure 4.3: Special Cases of the Dot Product of Non-zero Vectors: a) for $\varphi = 0°$ and b) for $\varphi = 180°$

The biggest advantage of this applet is that it makes the connections between the arithmetic-algebraic, the geometric and the structural mode of description and thinking transparent and easy to grasp. Namely, different positions of the Applet 1 show the axioms for the dot product of vectors. They are as follows.

1. Bi-linearity.

1a. Scaling (homogeneity property) is viewable on the Applet 1. Dot Product (Figure 4.4).

$k\left(\vec{u}\cdot\vec{v}\right) = \left(k\vec{u}\right)\cdot\vec{v}$

Scaling the vector $\vec{u} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ by factor 2 affects change of dot product from 8 to 16, thus also by 2, Figure 4.4.a) and b).

$k\left(\vec{u}\cdot\vec{v}\right) = \vec{u}\cdot\left(k\vec{v}\right)$

Scaling the vector $\vec{v} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ by factor 2 affects change of dot product from 12 to 24, thus also by 2, Figure 4.4.c) and d).

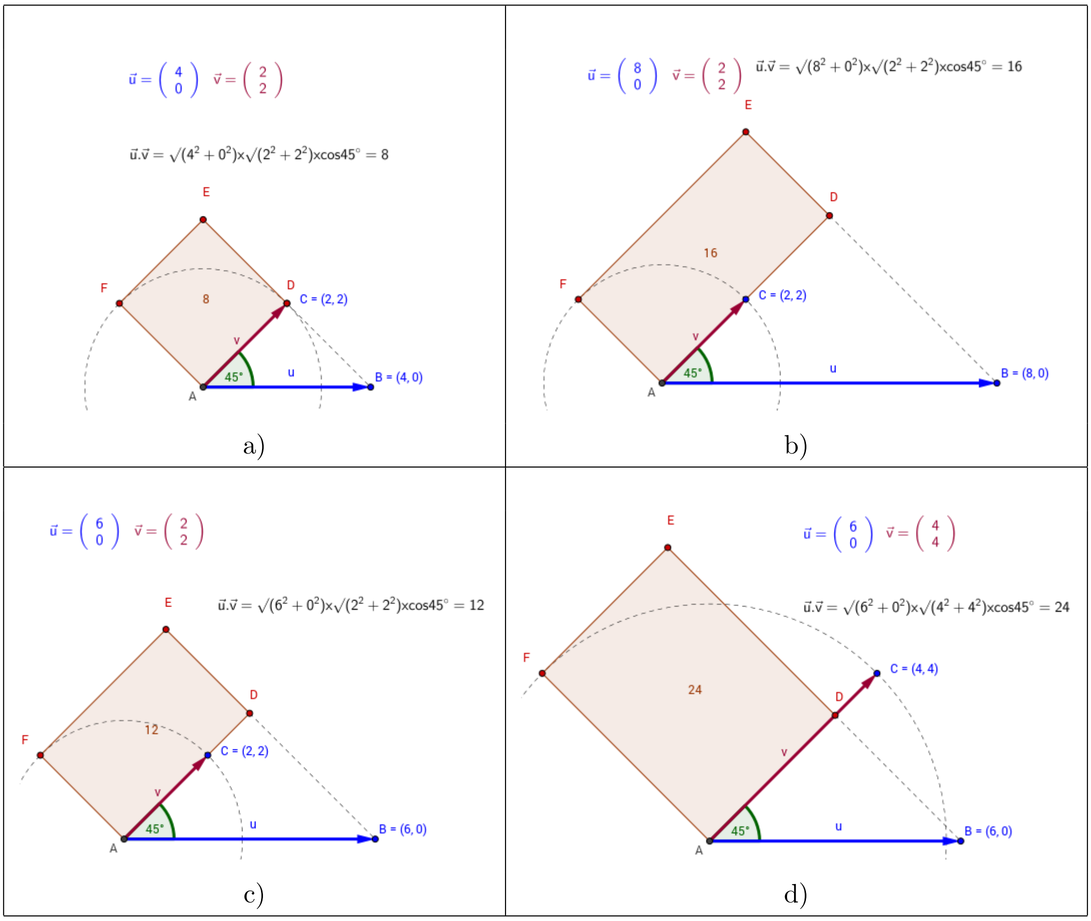


Figure 4.4: Applet 1 for the Axiom 1a. Scaling Property of the Dot Product

The homogeneity property[4] of the dot product may be investigated by changing the vectors (in magnitude, direction or orientation) or changing the angle between the vectors.

1b. Additive property of dot product is not viewable on this applet, but it may be visualized, also dynamically, with an additional **Applet 2. Additive Property of the Dot Product** (Figure 4.5). Compared to the previous applet, this one uses real numbers instead of integers. Having real numbers as inputs for the components of the vectors requires a considerable time amount on simple calculations. However, it is an additional outcome of this study and is available for dissemination (see Appendix A and Subection 6.2.2 of this thesis).

Focus on the properties 1a. homogeneity and 1b. additivity is important for developing the concepts of "linearity", "bi-linearity", as exemplified here by the dot product, and "multi-linearity", as exemplified by the determinants ([Donevska-Todorova, 2016b]).

[4]If using the applet is for an exclusive purpose of studying only this property, then including a slider for the scalar k, as in Applet 1, may be an advantage.

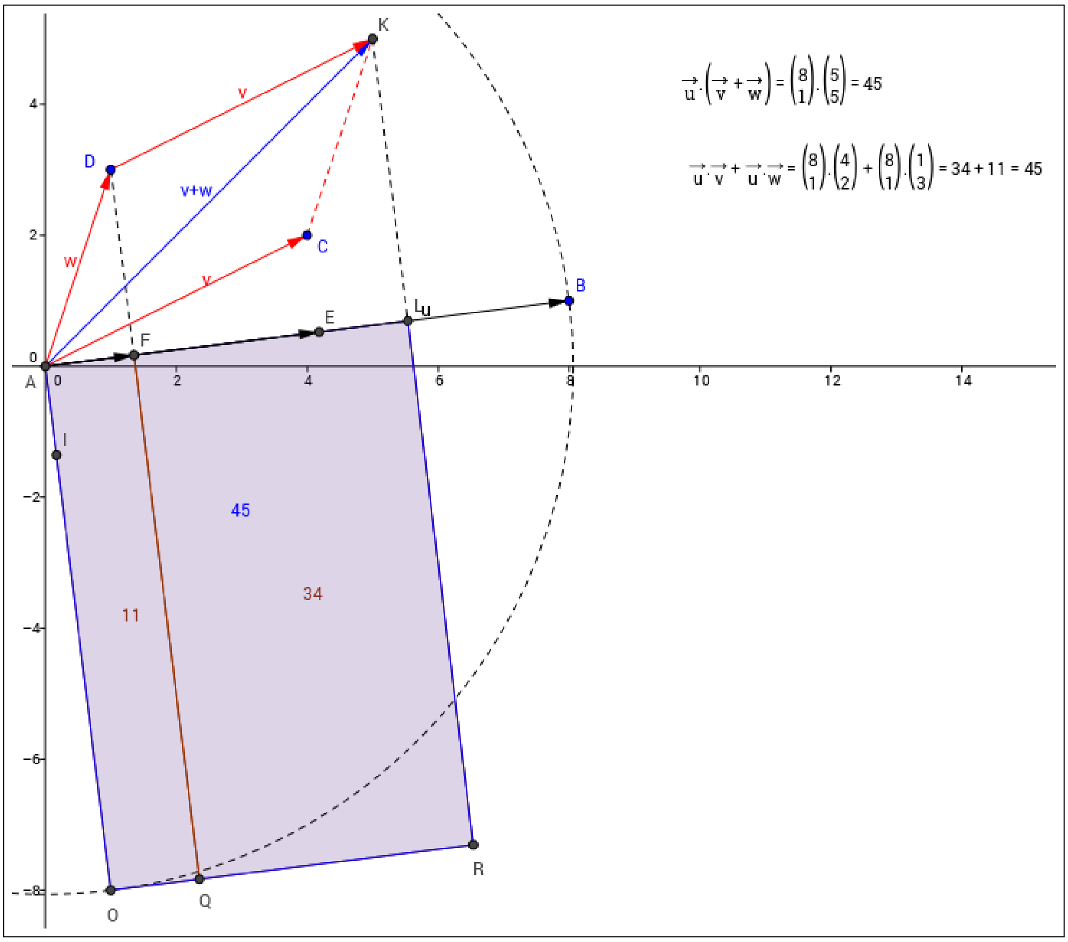


Figure 4.5: Applet 2. Additive Property of the Dot Product

2. Symmetry.

$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

$\vec{u} \cdot \vec{v} = |\vec{u}| \underbrace{|\vec{v}| \cos\varphi} = |\vec{u}| \left(\pm |\vec{v}_{\vec{u}}|\right)$ and $\vec{v} \cdot \vec{u} = |\vec{v}| \left(\pm |\vec{u}_{\vec{v}}|\right)$

For the learning of this property, instead of creating a new applet, I decided to offer a task in a paper-pencil environment. The major reason for such decision is for the students to use the opportunity to reflect on the experiences gained by the interactions with the Applet 1 and express their findings in another medium. I consider that, it is important that the students can register their own discoveries with the standard mathematical language using geometric and algebraic means by writing. The task is the following.

Task 1. Dot Product

Construct another quadrilateral whose area equals the dot product of the given vectors $\vec{u}$ and $\vec{v}$ on the Figure 4.6.

The last property constructing the axiomatic definition of dot product of vectors is the following.

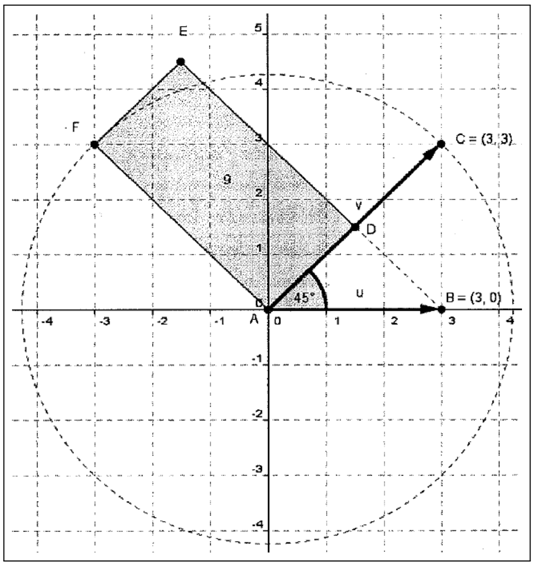


Figure 4.6: Task for the Visualization of the Axiom 2. Symmetry of the Dot Product

3. Positivity.

$\vec{u} \cdot \vec{u} \geqslant 0$, $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$

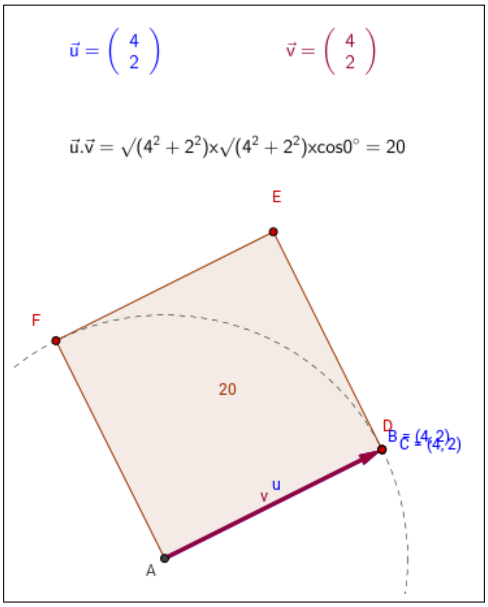


Figure 4.7: Visualization of the Axiom 3. Positivity of the Dot Product with the Applet 1

Since the angle between a vector $\vec{u}$ and itself is $0°$, the value of the cosine is 1. Hence, the cosine of the angle no longer has the influence on the $\pm$ sign of the dot product. Therefore, the oriented area of a rectangle becomes an area of a square, namely, $\vec{u} \cdot \vec{u} = u^2$ which is always positive. This property of the dot product can well be illustrated by the Applet 1. Dot Product (Figure 4.7).

It seems that the proposed Applet 1. and Applet 2., accompanied with the Task 1. Dot Product, have the potential to support the development of all three modes of description and thinking of the dot product.

4.2.1 Engagement with the Applets for the Dot Product

Students' actions when they are engaged in the DGE can be described by *Variational Dragging Scheme (VDS)* ([Leung et al., 2006]). The VDS for the Applets 1 and 2 by dragging modalities of points includes students' involvement in changing:

1. Coordinates of points
2. Components of vectors
3. Magnitudes, directions and orientations of vectors
4. Counter-clockwise or clockwise orientations of vectors
5. Projections of vectors
6. Angles between vectors (affecting the $\pm$ signs or zero value of the dot product)
7. Areas of corresponding rectangles

Dragging modalities for the Applet 1. and Applet 2. include not only changes of lengths and directions of vectors or changes of coordinates of points, but also changes of angles, although they can only implicitly be changed. Thus, the VDS of students' instrumented actions with these Applets in the designed DGE includes an action for focusing on the trigonometric function cosine of an angle. It is immediately visible on the Applet 1 that one side of the rectangle has the same length as the length of one of the vectors (the reason why the circle also appears dashed, on the applet). The length of the other side of the rectangle equals the length of the projection of the other vector over the first one, thus the product of the length of the second vector and the cosine of the angle between both vectors. Acute angles lead to positive values while obtuse angles lead to negative values of the dot product. For the case of an angle with 90° there is no rectangle, thus, the dot product is zero (Figure 4.2.b). These values of the dot product are displaced in the arithmetic-algebraic mode of description on the top of the Applet 1, while their absolute values appear on the corresponding rectangles (as their areas), thus on the geometric mode of description of dot product.

In conclusion of this section, the suggested approach with the Applets 1. and 2. using projections of vectors integrates the visual-geometric and the arithmetic-algebraic modes of description and tries to bring them in connection with the structural mode of description by illustrating axioms for homogeneity, additivity, symmetry and positivity of dot product of vectors in a DGE.

Findings of the application of the above design regarding the dot product gained during the actual learning trajectory are discussed in Section 5.2.

4.3 Suggested Approach for Determinants (ARQ3)

This section shows a possible approach for supporting the learning of the axiomatic properties of determinants (Table 4.1) by the use of two applets in a DGE: **Applet 1. Determinants** (Figure 4.8) and **Applet 2. Additive Property of Determinants** (Figure 4.9)[5]) (see also [Donevska-Todorova, 2012b], p. 116).

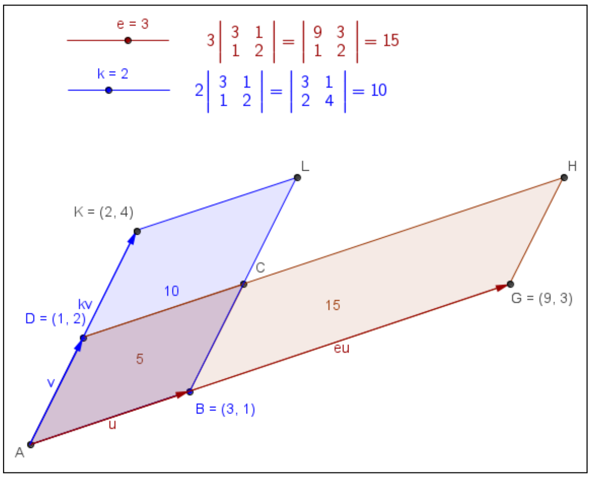


Figure 4.8: Dynamic Applet 1 for Axiom 3a

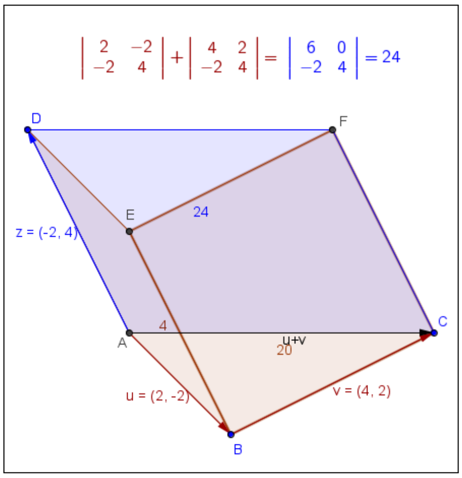


Figure 4.9: Dynamic Applet 2 for Axiom 3b

The transition between the high school and university approaches in defining determinants can be seen through the difference in notation[6] and language usage. Namely, the left column in Table 4.1 shows notation and symbolism which can be used in high school, while the right column in Table 4.1 shows notation which is typical for the university level of Linear algebra. There is, of course, also a difference in the level of abstraction. Namely, the school approach refers dimension 2 (or analogically 3), while the university approach refers to a generalization to dimension n[7].

[5]The geometric visualization on the Applet 2 shown in Figure 4.9 is similar to the static visualization shown in Figure 1.13, p. 30 in this thesis, but has a dynamic character, and moreover, a simultaneous appearance of the arithmetic and the geometric modes of description.

[6]The two vertical lines in the notation of determinants for secondary level approach should not be confused with absolute value ([Fischer, 2008], p. 179).

[7]The level of generalization and abstraction was discussed in Subsection 1.4.1 and is, further on, discussed in Subection 6.2.2.

Secondary Approach	**Tertiary Approach**
1. $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ **Applet 1** (Figure 4.10.a)	**1.** $\det E_n = 1$
2. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$ **Applet 1** (Figure 4.10.b)	**2.** $\det \begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_j \\ \vdots \\ r_n \end{pmatrix} = -\det \begin{pmatrix} r_1 \\ \vdots \\ r_j \\ \vdots \\ r_i \\ \vdots \\ r_n \end{pmatrix}$
3a. Homogeneity: **Applet 1** (Figure 4.8) $e \cdot \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} e \cdot a & e \cdot b \\ c & d \end{vmatrix}$ $k \cdot \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ k \cdot c & k \cdot d \end{vmatrix}$	**3a.** Homogeneity: $e \cdot \det \begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_n \end{pmatrix} = \det \begin{pmatrix} r_1 \\ \vdots \\ e \cdot r_i \\ \vdots \\ r_n \end{pmatrix}$
3b. Additivity: **Applet 2** (Figure 4.9) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix} = \begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix}$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ c' & d' \end{vmatrix} = \begin{vmatrix} a & b \\ c+c' & d+d' \end{vmatrix}$	**3b.** Additivity: $\det \begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_n \end{pmatrix} + \det \begin{pmatrix} r_1 \\ \vdots \\ r_i' \\ \vdots \\ r_n \end{pmatrix} = \det \begin{pmatrix} r_1 \\ \vdots \\ r_i + r_i' \\ \vdots \\ r_n \end{pmatrix}$ $e \in \mathbb{R}, r_1, ..., r_i, r_i', ..., r_j, ..., r_n \in \mathbb{R}^n$

Table 4.1: Axiomatic Properties of Determinants

The axiomatic definition which is suggested in the hypothetical learning trajectory (HLT) consists of axioms 1, 2, 3a and 3b (see Table 4.1, p. 84). The choice for this particular set of axioms was considered suitable for upper-secondary school (compared to D1, D2, and D3 in Subsection 3.1.3, p. 57). Other properties of determinants could also be chosen for Axiom 2, as for example $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$ ([Fischer, 2011], p. 278) which would not affect the existence and uniqueness of the determinant function. The geometric interpretation of the axioms refers to the area of the unit square in Axiom 1 and oriented areas of parallelograms in Axioms 2, 3a and 3b.

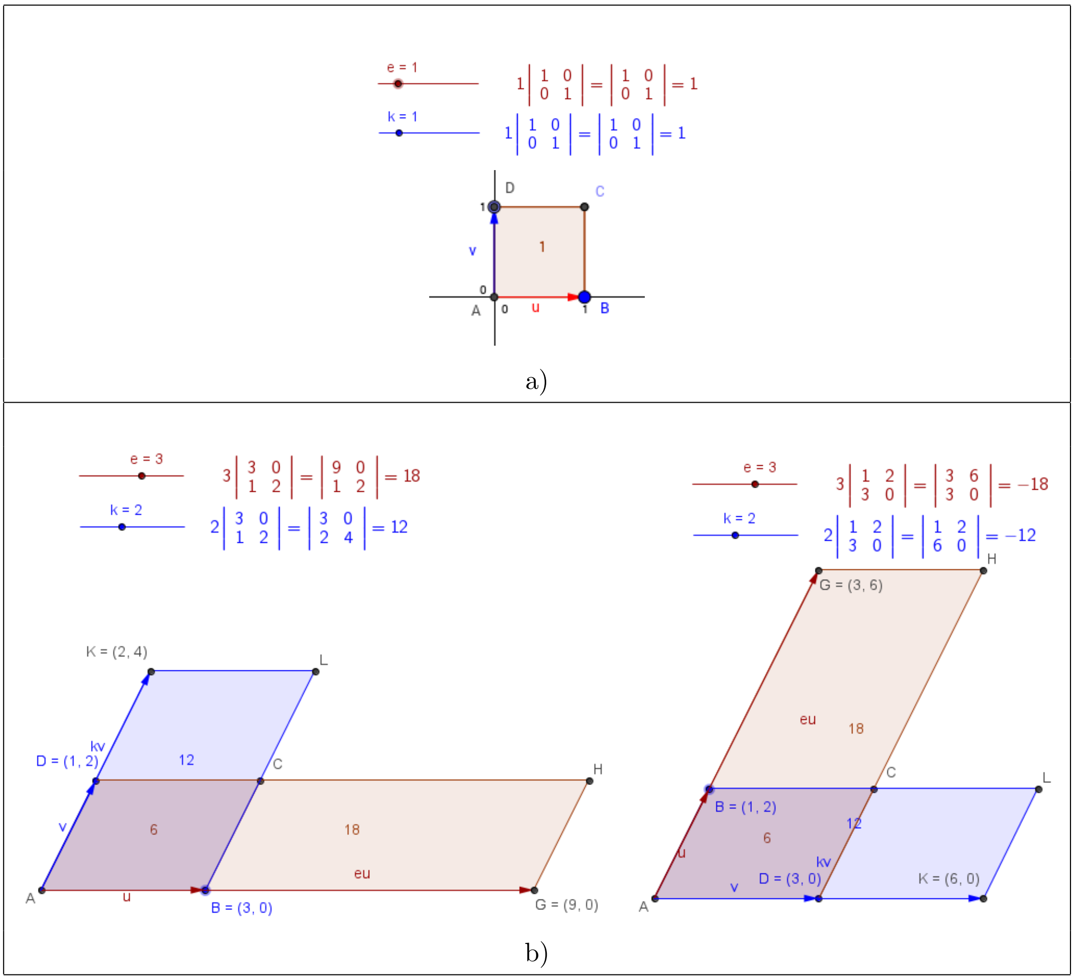


Figure 4.10: Different Positions of Applet 1 Demonstrating: a) Axiom 1 and b) Axiom 2

Discussing Questions for learning determinants with the applets in the proposed DGE aim to support concept definitions and concept images of determinants by connecting them to vectors and plane geometric figures. The questions are organized into two groups, one addressing the Axiom 3a in relation to the Applet 1 (Figure 4.8), and two, addressing the Axiom 3b in relation to the Applet 2 (Figure 4.9).

Discussing Questions for the Axiom 3a and Axiom 1 with Applet 1

1. Double the length of one side of the parallelogram $ABCD$. (Set one of the sliders at 2 and the other one at 1).

a) How does it affect its area?

b) Write your answer with the notation of determinants.

c) Compare the length and the direction of the vectors $\vec{u}$ and $e \cdot \vec{u}$ (or $\vec{v}$ and $k \cdot \vec{v}$). What is their relation to the determinant?

2. Double the lengths of both sides of the parallelogram. How does it affect

   a) its area?

   b) the entries in each row of the determinant?

   c) the value of the determinant?

3. Double one of the sides of the parallelogram and triple the other one.

   a) How does it affect its area? Write your answer with the notation of determinants.

   b) Compare the result with that of the previous task.

4. Explore the situation for other real numbers and generalize your answer.

5. Set both sliders at 1, B at $(1, 0)$ and D at $(0, 1)$.

   a) Which geometric figure do you obtain?

   b) Which of the axioms for determinants is shown?

The first four questions in this first group of discussing questions aim to develop the geometric mode of description and thought with simultaneous attention to bi-linearity of the determinant function, in particular homogeneity (by scalar multiplication of the vectors on the Applet 1, Figure 4.8), while the discussing question 5. is a special case of the determinant of the unit matrix, i.e. Axiom 1 (Figure 4.10 a). Further on, this Applet 1 can be used for exploring the orientation (Axiom 2), by comparing the two positions on Figure 4.10 b), for example.

Discussing Questions for the Axiom 3b with Applet 2

1. What is the relation between the vectors $\vec{u}$, $\vec{v}$ and $\vec{z}$, and the determinants?

2. Explain the relation between the areas of the parallelograms on the applet and the determinants.

The first question in this group addresses the bi-linearity, in this particular case the property of additivity (in a column/ row) of the determinant function in connection with vector addition. The second discussing question connects determinants with oriented areas of the three parallelograms presented on the Applet 2 (Figure 4.9). Screen-shots of both Applets 1 and 2 together with the discussing questions as they appear on GeoGebraTube can be seen in Appendix A of this thesis.

Homework problem for the Axiom 1 in 3D

1. Sketch the unit cube $ABCDA_1B_1C_1D_1$ in the $3D$ Coordinate System, placing the vertex A at the origin.

2. Using the sketch, fill in the blanks.

   a) The volume of the unite cube $ABCDA_1B_1C_1D_1$ is: ————–.

   b) The coordinates of the vertices of the unite cube $ABCDA_1B_1C_1D_1$ are:

   ____

   c) The determinant representing the volume of the unit cube, using the coordinates of its vertices is:

   ____

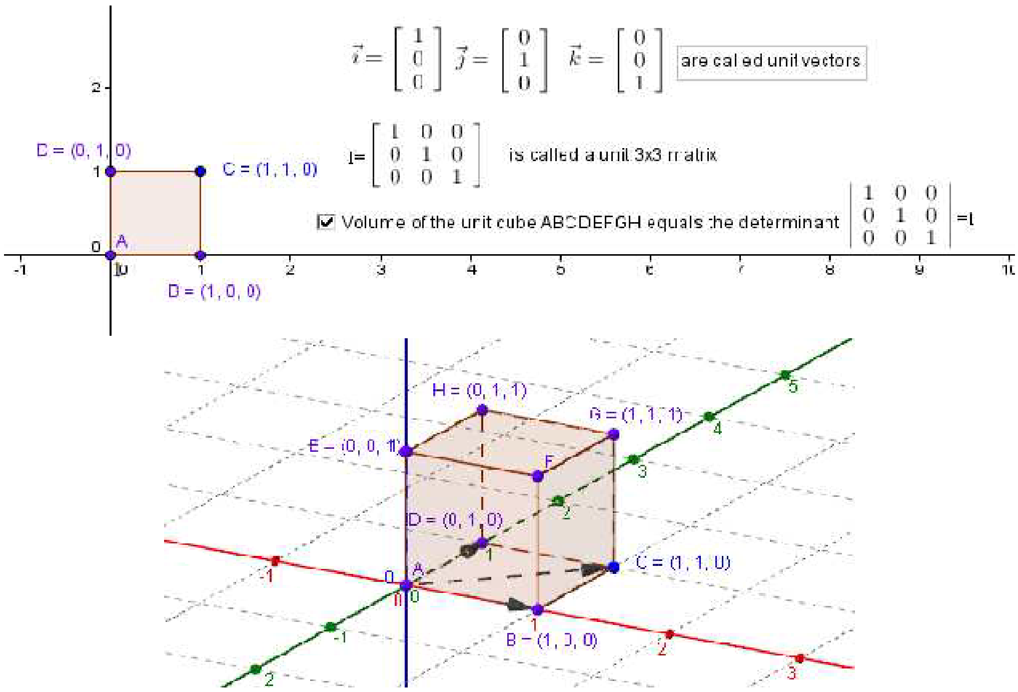


Figure 4.11: Static Visualization of Axiom 1 for Determinants in Order Three

Students have to sketch the unit cube, to calculate its volume and to denote its vertices in a rectangular coordinate system. The main idea is to provoke students' autonomous conclusions that this small set of tasks refers to Axiom 1 in the definition of determinants (see Table 4.1), but in order three. From a global point of view, it aims to establish a base for generalizations in dimension n at the tertiary level of learning Linear algebra.

Besides the visualization of the Axiom 1 in 3D, the static visualization on Figure 4.11[8] integrates the geometric mode (vectors in 3D coordinate system) and the arithmetic-algebraic mode (vectors as ordered triples) of description. Thus, the unit vectors, unit matrix, and its determinant are visible in both of these modes of description.
Students' solutions on of this small set of homework problems are shown in Subsection 5.3.3).

4.3.1 Integration of the Three Modes of Description for Vectors and Determinants in the DGE

In this section, I explain how all three modes of description and language for vectors and determinants (see Section 2.3) are related to each other by the aid of the technology-based design in this project. I start with a detailed description of the integration of the three modes of description for vectors and continue with determinants, including technical considerations about the applets.
Both Applets 1 and 2 (Figure 4.8 and Figure 4.9) are interactive dynamic visualizations combining *three languages and modes of description*[9] ([Hillel, 2000]) both for vectors and determinants in one interface.
First, they show the *geometric mode of description* (1-, 2- and 3- dimensional spaces) of **vectors** represented by directed line segments (arrows) starting from a common point (the origin), having magnitude and direction, labelled as $\vec{u}$, $\vec{v}$ and $\vec{z}$ on the Applets. This notation characterizes the *geometric language* distinguishing between vector and scalar quantities. The operation "vector addition" is defined by the parallelogram rule (head and tail rule is also viewable) and the operation "scalar multiplication" is defined by stretching/ shrinking of a vector (by dragging its ending point). It must be underlined that all initial points of the vectors on the applets are fixed at a point (origin) and are representatives of equivalence classes of vectors. This crucial issue has its particular value in applying vector translations in Task 2. Triangle and Task 3. Trapezoid (given in the HLT in Section 4.4), which helps to bring determinants in connection with areas of triangles and trapezoids, respectively. This synthetic (coordinate-free) geometric mode of description can easily be transformed into the analytic (coordinate) geometric mode of description by simple selection of the "show axes" command on each of the applets' surfaces in the DGS. The DGS also provides a grid of the Cartesian coordinate system with standard orthogonal axes and equidistant units on each of the axis, but non-uniform units

[8]Remark: Due to the restricted possibilities of the dynamic software GeoGebra which was used at the moment of the creation, this visualization of the Axiom 1 in 3D instead of dynamic (as in the previous applets) possesses static characteristics. Further upgrade of the suggested design in this project may benefit by analogically following the same idea for 3D applets to the offered ones here.

[9]Although some literature name geometric and algebraic mode of description as representations, [Hillel, 2000] makes a clear distinction between modes of description and representations and this research study utilizes his terminology, as was previously discussed in Section 2.3.

along both axes are also available which may further on serve as a precursor to the basis notation. When choosing this coordinate geometric mode, it is preferable that both unit vectors (standard basis) are shown on the applets in order to avoid future obstacles with changes between the standard and non-standard basis in correlation with Hillel's claim (2000, p. 197). The operation "addition" in this mode of description, is defined in terms of coordinates of the terminal point of the diagonal vector in the parallelogram equal to the sum of the corresponding coordinates of the terminal points of vectors on the parallel sides of the parallelogram (viewable on the applets).

The second, *algebraic mode of description* consists of vectors represented as n-tuples of real numbers $(u_1, u_2, ...u_n)$ labelled by capital letters and is available in the algebra window in the DGS. This notation is also known in the literature as *algebraic language*. The operations "vector addition" and "scalar multiplication" can be defined component-wise at the beginning of the teaching sequence, based on the field properties of the set of real numbers as an upgrade of students' previous mainly geometric knowledge in mathematics and physics. This mode of description is developed from vectors as ordered pairs in two dimensions to vectors as ordered triples in three dimensions (utilizing the DGE) and thus has the potential to be generalized for vectors as ordered n-tuples in $\mathbb{R}^n$.

The third, *abstract mode of description* of vectors as elements of a vector space is restricted only to initial knowledge. Namely, during this study, I designed two additional applets for the commutative and associative properties of vector addition (see [Filler & Donevska-Todorova, 2012] and Appendix A, p. 174). They are considered as an additional contribution of this doctoral project and may be used for further research (see Subsection 6.2.2).

Both Applets 1 and 2 (Figure 4.8 and Figure 4.9), as a part of the designed artefact, have further values. Namely, they interlink the *geometric mode of description* of **determinants** as oriented areas (volumes) of geometric figures (solids) in particular parallelograms (parallelepipeds) in two (three) dimensional Euclidean geometry, with the *algebraic mode of description* of determinants (as functions of quadratic 2×2 (3×3) matrices with real values), both visible on each of the applets. They are also valuable for the *abstract mode of description* of determinants because they support all definition axioms for determinants, 1, 2, 3a and 3b in the created artefact, without which the whole axiomatic approach at the secondary education seems extremely difficult, if not impossible.

The usefulness of the applets, for interchanging all three modes one into another, is due to the features of the dynamic interface to integrate more modes in one place. It is the DGE which allows subtle movements between *geometric-algebraic* and *algebraic-abstract* mode of representations ([Hillel, 2000]). The geometric-algebraic mode of representation, by its nature, connects the geometric with the algebraic mode of description and it shows that "parallelogram vector addition" is compatible with component-wise vector addition ([Hillel, 2000]), and moreover, vector addition

and scalar multiplication in terms of parallelograms (stretch/ shrink of a side) are compatible with linearity of determinant function in a row (column) in terms of real number entries of a square matrix. The algebraic-abstract representation secures integration of the algebraic mode of description into the general axiomatic treatment of the unifying theory. These shifts between the modes of description allow a start of the intended teaching/ learning sequence with the geometric mode and then back and forth translations between all modes of description and languages, as undertaken during the actual learning trajectory (ALT) and experiments (Section 5.1).
Since the Hillels' modes of representations of concepts are fundamentals for the *three modes of thinking* in Linear algebra, *synthetic-geometric*, *arithmetic-algebraic* and *analytic-structural* ([Sierpinska, 2000]), the artefact and the instrumentation have potentials to support mathematical thinking. Namely, the abstract mode of description corresponds to the analytic-structural mode of thinking. In this sense, the proposed approach for visualizing the axiomatic properties dynamically contributes to an easier learning of the general formal theory of Linear algebra. In comparison with the arithmetic-analytic mode of thinking, it is crucial to be distinguished between different levels of abstraction. Both modes of thinking may lead to a correct solution, but they substantially differ in the level of abstraction with dominance on the side of the analytic-structural thinking. When recalling and applying particular axioms of the definition students develop a wider picture on how the whole Linear algebra theory bases on the importance of definitions and theorems. This fact is very relevant, for example for proving.

A look back at the suggested approach for the teaching and learning determinants in this suggested DGE (Section 4.3 and Subsection 4.3.1) shows that it considers all three modes of description and thinking with a particular focus on each of the defining axioms.

4.3.2 Engagement with the Applets for Determinants

After a very short time, the paper-pencil technique for calculating determinants in order two (and also in order three, with the rule of Sarrus or the Leibniz formula) becomes a routine for the students. In order to avoid such routinized work, these calculations are embedded in the applets, as they directly show values of all displaced determinants. Thus, these values have a justificatory and explicative character, for example by their comparison with the displaced areas of the geometric figures. In this way, the *epistemic value* of the paper-pencil technique for calculation of determinants becomes a *pragmatic value* of the techniques in the technology environment ([Artigue, 2002]). The by-hand skills for writing are substituted by new epistemic values of instrumented gestures as techniques for connecting different modes of description, for determining the plus or minus sign of determinants, for visualizing their properties, for structurally connecting more concepts, etc. These epistemic

values of both applets are explained in details through two *Variational Dragging Schemes (VDS)* ([Leung et al., 2006]).
Exact steps of students' actions which could be identified as elements of a Variational Dragging Scheme (VDS) by dragging of sliders and points, while students work with the **Applet 1** in the created DGE are:

1. Focusing on the change of length and opposite (but not any other) direction of a vector by dragging a slider (when it represents a positive or a negative number/ scalar) and thus recalling linear (in)dependence of vectors.

2. Reasoning how dragging a slider affects change of length of a side of the given parallelogram and simultaneously its area.

3. Detecting how dragging a slider affects the coefficient multiplying a determinant which consequently affects the final result in the algebraic representation by experiencing the linearity of the determinant function in a row (Axiom 3a).

4. Attempting to generalize Axiom 3a by experiencing with the other slider.

5. Creating contrasting experiences on the change of the plus or minus signs of determinants, by dragging the terminal points of the vectors in a clockwise or counter-clockwise direction in each quadrant of the Cartesian coordinate system (Axiom 2).

6. Simultaneously experiencing changes of the area by dragging the vertices of the parallelogram in each quadrant of the Cartesian coordinate system and focusing on the absolute value of the determinant.

7. Recalling vector addition and linear combinations of vectors by dragging the terminal points of the vectors.

8. Generalizing findings referring to vectors and determinants for any real numbers and any positions of the points on the plane.

9. Attempting to generalize (hypothesizing) both algebraic and geometric findings for vectors and determinants in order three (and n).

Similar VDS of students' instrumented actions when exploring the **Applet 2** in the DGE, with an exception that this applet does not utilize sliders, is:

1. Focusing on the change on length, direction and orientation of vectors by dragging their terminal points, thus recalling linear (in)dependence of vectors.

2. Referring to vector addition (also possible check of commutative property) and linear combinations of vectors by dragging terminal points of the vectors.

3. Simultaneously experiencing changes of the area by dragging the vertices of the parallelogram in each quadrant of the Cartesian coordinate system and focusing on the absolute value of determinants.

4. Establishing connections to proving in elementary geometry through investigations of different positions of the vertices of the formed parallelograms and triangles (using corresponding parallel sides and equal angles to justify congruence of triangles by SSS, SAS or ASA).

5. Discovering additive property of the determinant function (Axiom 3b) by dragging the terminal points of the vectors affecting entries in a row.

6. Generalizing the additive property of the determinant function for any real numbers in two dimensions and attempt for generalizing in three and n dimensional space.

These dragging modalities and strategies, whether of points or of sliders in the applets are observed as dynamical tools which enable simultaneous changes of algebraic and geometric entities. The combination of these two specific technological tools, movable points, and sliders, may bring dynamic character to the learning process of concepts in the focus, by keeping the students engaged. With their aid, students have the opportunity to see how translations from one into another mode of description for vectors and determinants occur in a natural and smooth way and thus, can comprehend and construct own translations. These interactions between the artefact and the students shape the learning trajectory ([Sacristán et al., 2010]). They no longer have to remain to shape the hypothetical learning trajectory, but may go beyond and construct the actual learning trajectory which may lead to deeper conceptual understanding ([Sacristán et al., 2010]) by combining and translating between multiple modes of description and language and establishing structured links between many concepts. The sliders, as contributory tools that fashion the learning process in a DGE, may show to be relevant for enhancing and complementing knowledge on real numbers, vectors, and determinants. So the technical aspect of the physical actions of dragging may have its own educational value as also exemplified by both Variational Dragging Schemes and they may co-emerge to build a powerful medium for students' progress in learning.

4.4 Tasks for Investigating Conceptual Understanding of Determinants through Translations between Three Modes of Descriptions in Problem Solving (ARQ4)

The HLT in this section continues with a set of four tasks[10]. They aim to investigate how students use their previous knowledge about plane geometric figures and vectors when learning a new topic about determinants. Besides the calculation of a given determinant students have to explain, draw, prove and write formulas about areas and determinants in order two in general terms.

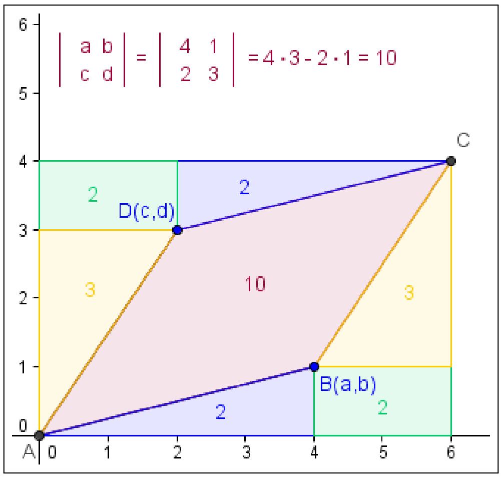


Figure 4.12: Applet 3 for Area of a Parallelogram with Determinants

After interactions with the interactive dynamic Applet 3[11] (Figure 4.12) students have to transfer their new knowledge into a paper-pencil environment by solving Task 1. Parallelogram, Task 2. Triangle, Task 3. Trapezoid and Task 4. Locus.

Task 1. Parallelogram

a) Calculate the determinant and give a geometric interpretation. $\begin{vmatrix} -4 & 4 \\ 3 & -1 \end{vmatrix}$.

b) Provide a geometric figure in the Cartesian coordinate system to validate your answer in a).

c) Provide an arithmetic-algebraic argumentation to validate your answer in a).

While Task 1 represents a standard problem connecting determinants with oriented areas of parallelograms, Tasks 2 and 3 represent its variation. It is a variation which

[10] A worksheet containing this set of tasks which was given to the students is available in Appendix D in complete.

[11] Similar static visualizations can be found in research literature and textbooks (e.g. [Tietze, Klika & Wolpers, 2000], p. 212, see Figure 1.11, p. 27 in this thesis, and [Fischer, 2011], p. 284).

requires a step further in both algebraic and geometric thinking because it connects determinants with geometric figures which are not parallelograms, but triangles and trapezoids.

Task 2. Triangle

a) Sketch the triangle ABC with $A(5,0)$, $B(-1,4)$ and $C(-3,-2)$ in the Cartesian coordinate system.

b) Calculate the area of the triangle ABC using determinants.

c) Write a formula for the area of a triangle ABC with $A(a,b)$, $B(c,d)$ and $C(e,f)$ using determinants.

Task 3. Trapezoid
Find the area of a trapezoid $ABCD$ given with coordinates of its vertices using determinants.

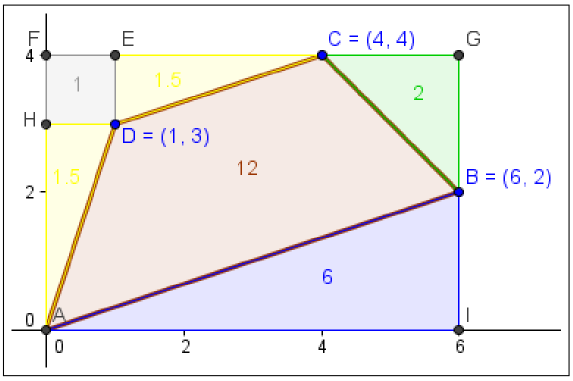


Figure 4.13: One Possible Solution of the Task 3 Constructed with GeoGebra

The HLT ends with Task 4. Locus and Applet 4 (Figure 4.14). With these created materials in the DGE students can upgrade their knowledge about the point-slope form $y - y_1 = m(x - x_1)$ and the slope-intercept form $y = kx + m$ of the equation of a line, by connecting it to the formula for the calculation of the area of a triangle $A = \frac{1}{2}ah$, within a new context of determinants in order 3.

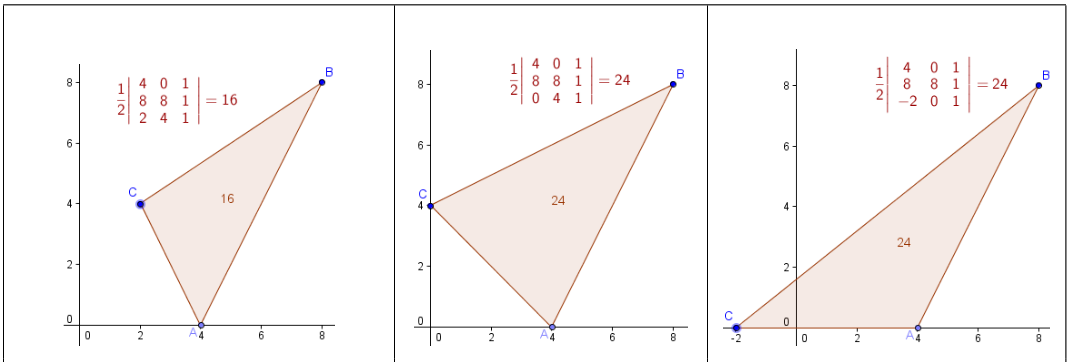
Task 4. Locus
Given the vertices $A(4,0)$ and $B(8,8)$ for a triangle ABC. Find the vertex C such that half the absolute value of the determinant is 24:

a) if $c \in y-$axis

b) if $c \in x-$axis

c) Which equations must all these vertices C satisfy?

d) What is the locus of the point C?



Initial Position of the Applet 4 | Solution of the Task 5a) | Solution of the Task 5b)

Figure 4.14: Different Positions of Applet 4

In order to solve this Task, students must recall their previous knowledge about concepts in Analytic geometry as a point on a line, intersections of two lines and collinearity, and concepts in Elementary geometry as areas of triangles.

The actual learning trajectory (ALT) regarding these four tasks is given in Section 5.4.

In summary of this chapter, the careful design of the whole HLT consisted of applets, tasks, discussing questions seems to have the potential to support students in deep conceptual understanding by trying to integrate not only the geometric and algebraic, but also the axiomatic mode of description of the dot product and determinants. In particular, taking input knowledge about *vectors* and *linearity* (Preliminary Study, p. 4.1), it aims to develop understanding of the:

- *bi-linearity of the dot product* with the aid of:
  - **Applet 1. Dot Product** (Figure 4.1, p. 75) and
  - **Applet 2. Additive Property of the Dot Product** (Figure 4.5, p. 80)

  and

- *multi-linearity of determinants* with the aid of:
  - **Applet 1. Determinants** (Figure 4.8, p. 83) and
  - **Applet 2. Additive Property of Determinants** (Figure 4.9, p. 83).

This is in close connection to the Research Problem (p. 61) in this study.

Chapter 5

Analysis and Results

This chapter discusses the **actual learning trajectory (ALT)** and the findings of the study. Indeed, the final teaching process results from a dialectic between the original design in the hypothetical learning trajectory and its realization in the classroom, what is usually an iterative process of development ([Ruthven et al., 2009]). The main focus of the analysis (Sections 5.1 to 5.4), according to the research questions, is on the *cognitive level* of the multiple levels of data analysis (p. 67). An analysis regarding the *interpersonal, classroom and resource leve*l is undertaken in Section 5.5.

5.1 Analysis and Results Regarding ARQ1

On the beginning of the HLT (Section 4.1), I have stated two questions about investigations on students' concept definitions and concept images of vectors (see Appendix C. Introductory Survey for the Preliminary Study, p. 187). This Pre-study (Phase 3 in the design cycle, Figure 3.2, p. 65 and Table 3.1, p. 66) should provide answers to the first auxiliary research question (ARQ1). The aim of this Pre-study is not to introduce vectors as part of the proposed teaching/ learning sequence in the course Linear algebra and Analytic geometry. Introduction of vectors has already been undertaken five months prior the actual learning trajectory (ALT) took place. Students' knowledge prior the experiment took place also includes: vectors (representations in a coordinate system, operations addition, scalar multiplication and dot product, linear combinations of vectors, linear dependence and independence of vectors), systems of linear equations, matrices and Gauss eliminations. The goal of the Pre-study is rather to collect information about the current stage of students' knowledge about vectors (concept definitions and concept images of vectors in Section 2.2, p. 40 and three modes of description and language for vectors in Section 2.3, p. 43). Collected information from the Pre-study are considered relevant for the influence on later constellations as for example, concept images of vector as classes of arrows and dominance of geometric modes of description and language for vectors suggest geometric introduction (definition) of the dot product; or concept definitions and concept images of vectors as n-tuples precondition an arithmetic-algebraic addition of determinants. This kind of constellations and the importance of consistency in description and language when introducing new concepts are sometimes unconsidered by textbooks authors (as was elaborated for example for vectors and the dot product in Section 1.4) and, as a consequence, may also be forgotten by the teachers. The Pre-study has one more, even more important, contribution. It enables students a possibility to refresh their previous knowledge and, by recalling basics about the concepts, to deepen their episodic memory or substitute it with long-lasting memory. This relies on the Harel's first indicator for conceptual understanding in connection with developing effective concept images, "remembering instead of memorizing" ([Harel, 1997], p. 109).

5.1.1 Analysis Regarding Vectors

The first question in the HLT, what is a vector, p. 74, in this small-scale Preliminary-study asks the students for a formal definition of vectors. This question was meant to evoke students' associations with symbolic notations of vectors, but above all it was expected that the answers may include both the algebraic and the geometric modes of description of the vectors. An alternative question 1 asking for an explanation of the vector concepts to a classmate was also offered to the students. It aimed evoking students' concept images of vectors. To support my interpretations of the obtained data, in the next paragraph I have inserted excerpts of the students' answers.

Question 1. (p. 74). All students worked on the first question, what is a vector. The responses reveal to distinguishing **two main groups** of answers, and **two categories** and **four subcategories** in the second group of answers, in order to be structured according their quality.

Group 1.

Students, who could not provide the formal definition, decided to answer the *alternative question.* Students in this group use quite vague formulations to express their ideas. I illustrate this statement by the following quotes from the students' answers (see Table 5.1).

These students' answers utilize phrases as: "I would explain it" (S1), "looks like", "has a" (S2), "specifies" (S5) and "symbolized by" (S6) which refer to students' internal representations, thus concepts' images rather than formal concept defini-

S1	Ich würde es so erklären, dass es ein Pfeil in einem koordinatensystem ist, welcher zwei Punkte verbindet. Translation of S1: I would explain it, that it is an arrow in a coordinate system, which connects two points.
S2	- looks like an arrow - has a certain ~~x~~ length and angle
S3	▷ Vektor ist durch Betrag und Richtung gekennzeichnet ▷ definiert Bewegung im Raum Translation of S3: A vector is identified by a magnitude and direction. defines motion in space.
S4	Mit einem Vektor kann man eine Bewegung im Raum darstellen. Translation of S4: A motion in space can be represented by a vector.
S5	Vektor zu geben Betrag und Richtung einer Einheit an. Translation of S5: A vector specifies magnitude and direction of a unit.
S6	- symbolized by an arrow, which shows the direction of the movement of a point

Table 5.1: Group 1 of Students' Answers on Question 1 in the Introductory Survey

tions. These descriptions are mostly driven from students' concept images which base on visualizations or concept's applications. Three of these answers point out "an arrow" (S1, S2 and S6) instead of "a class of arrows" which is a typical students' misconception for vectors. These answers show that students' concept images are mostly built on entities as: "an arrow" (S1, S2 and S6), "points" (S1), "length and angle" (S2), "a coordinate system" (S1), "space" (S3 and S4), thus geometric modes of descriptions. Terms as "motion" (S3 and S4), "magnitude and direction" (S3 and S5), "unit" (S5) and "movement" (S6) suggest students' concept images of vectors evoking from concepts in physics, where a vector is a quantity defined by a magnitude, direction and orientation, or eventually from mathematics textbooks in which vectors are introduced through translations.

Group 2.

This group is consisted of answers of those students who tried to provide a formal definition. Answers of the students in this group are further on categorized. This categorization uses the one suggested by [Rösken & Rolka, 2007]), who investigated the concept image and concept definition of integrals. My analysis includes additional comparison of answers to both questions provided by a single student. I made a similar categorization for the analysis of the students' answers to the second question in the Pre-study. Categories and subcategories in this Group 2 of the students' answers to the first question are defined depending on three indicators: one, "a class of arrows", two, "same orientation" and three, "same length".

These students' answers refer to a formal concept definition of vectors. An important attribute to these definitions is the assertion of "a class" and for this reason they are classified as *appropriate definitions* in G1 (see Table 5.2). Answers clearly stating the first indicator, that a vector is "a class of arrows" are classified in the subcategory G1a (S7 and S8). Answers lacking this precision and showing some students' doubts are classified in the subcategory G1b (S9 and S10). Although the *definiendum* in the definition S10 is omitted, it is interesting that the student recalled that "all parallel vectors with the same length" are equal, which match the image of a class of arrows. All definitions in G1 contain the other two indicators.

The second category of the students, answers is G2, consisted of *inappropriate definitions*, further on also divided into two subcategories G2a of *incomplete* and G2b of *incorrect* definitions. Definitions referring to a single arrow, thus lacking the word "a class" - the first indicator, are classified in the subcategory of incomplete definitions G1b (S11 and S12). Definitions of S13 and S14 are considered as incorrect, because they address to a segment instead of an oriented segment. Thus, it is the *definiens*, "a segment" in (S12) or "a distance" in (S13) which may be considered incorrect, because of the student's uncertainty in distinguishing between vector and scalar quantities. Definition of S15 refers to an ort vector of a point. The second part of this student's definition shows his/her misconception of a point "not being on a plane, but in space".

Similarly as the answers in group 1, all answers in both categories G1 and G2 in group 2, refer to many visual representations of the concept by revealing to: "an arrow", "a segment", "length", "direction", "parallel", "a point", "a plane", "space" and "coordinate system", which means that students recalled their thriving geometrical modes of description and thought as parts of their concept images. Two students (S9 and S11) even provided sketches.

One more category was imagined to be constituted of definitions referring to the *algebraic mode of description* of the concept. Two of the students provided algebraic representations: $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ (S11) and similarly $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (S12), but none of the students offered a formal algebraic definition of a vector being an ordered

Cath.	Sub-cath.	Students' Quotes	
G1 Appropriate Definition	G1a	S7	a class of arrows, which are parallel, have the same direction and the same length
		S8	clases of arrows, which are parallel, going in the same dirgetion and are equae in length
	G1b	S9	An arrow with a ~~certain~~ determined length and direction. direction length It's a class of arrows. (Pfeilklasse)
		S10	A vector is defined by its direction and length. It can be located anywhere, all parallel vectors of the same length and „pointing direction" are equal.
G2 Inappropriate Definition	G2a Incomplete Definition	S11	2D/3D: Ein Pfeil mit definierter Länge und Richtung: Wird dargestellt als: $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ bzw. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ → auch in höheren Dimensionen möglich... Translation of S11: 2D/ 3D: An arrow with defined length and direction, is represented as: $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ i.e. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ → also possible in higher dimensions ...
		S12	Ein Vektor ist ein Pfeil, welcher die Bewegung eines Punktes in einer Ebene „$\begin{pmatrix} x \\ y \end{pmatrix}$" oder Raum $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ beschreibt. Translation of S12: A vector is an arrow, which describes the motion of a point on the plane $\begin{pmatrix} x \\ y \end{pmatrix}$ or in the space $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.
	G2b Incorrect Definition	S13	-Strecke zwischen zwei Punkten auf einer Ebene, oder im Raum, welche eine Verschiebung beschreibt. Translation of S13: A segment between two points on a plane or in a space which describe motion.
		S14	- Strecke bzw. Abstand zwischen zwei Punkten in einer Ebene oder im Raum, welche eine Verschiebung beschreibt Translation of S14: A segment, for example, distance between two points on a plane or in space which describe motion.
		S15	A vector is a arrow from the origin to a point in the coordinate system. The point is not in a plane. The point is in space.

Table 5.2: Group 2 of Students' Answers on Question 1 in the Introductory Survey

n-tuple (ordered pair or ordered triple) of real numbers (except the sort note "also possible in higher dimensions" by the S11), nor of a vector being an element of a vectors space. This fact was expected, having in mind the curricula and the way in which the concept of a vector was introduced at this level of education.

Overall, students' answers to the Question 1 show that the geometric mode of description of the concept is more emphasized in students' concept definitions and moreover, it is strongly linked to the image of an arrow, like those on the visual representations of S9 and S11. This conception proves to be rather demonstrative and is memorized by the majority of the students.

I continue to investigate if the students develop an adequate understanding of the concept of linearity.

5.1.2 Analysis Regarding Linear Combinations of Vectors

Question 2 (p. 74). Almost half of the students tried to answer the second question, what is a linear combination of vectors. The concept of linear combinations of vectors is important for linear dependence and independence of vectors, and in this study it aimed facilitating the leaning of bi-linearity of dot product of vectors and multi-linearity of determinants (for further reading see [Donevska-Todorova, 2016a]). For the interpretation of students' answers, they are again organized into two main categories as presented in the Table 5.3.

Analytic definition of linear combination of vectors uses symbolic notation with particular accent on the distinction between vector and scalar quantities. The use of analytic-algebraic language is evident in students' answers S10 and S16, but is omitted in the rest of the answers. These two answers use correct notations and clearly distinguish between scalar and vector quantities. Therefore they are classified as an *appropriate definition* and a *good attempt*, respectively. The definition of S16 does not specify that r and s are scalars, but uses the term scalar multiple. Definition by S16 even states that the result is a vector. Both definitions S10 and S16 refer to a linear combination of exactly two vectors. It is a matter of assumption whether these students are also able to generalize the definition for a linear combination of more than two vectors.

Students' answers S6 and S12 represent only special cases for parallel vectors, thus are limited. Furthermore student S12 used an improper notation for the vectors, witting v_1 and v_2 (S12 in Table 5.3) instead of $\vec{v}_1$ and $\vec{v}_2$ which lead her/him to a confusion in distinguishing between scalar and vector quantities (S12 wrote: "v_1 and $v_2 \neq 0$", and also "v_1 and $v_2 \neq 1$" in Table 5.3). It may also be argued that the student S12 refers to linear dependence and independence of vectors. Although incomplete, these students' concept definitions and concept images (S6: "v_1 and v_2 are linear combination one from another when $v_1 = kv_2$" and S12: "both vectors are having the same direction (parallelism)") are very important for the homogeneity

Cath.	Sub-cath.	Students' Quotes	
A1 Appropriate Definition		S10	$a\cdot\vec{x}+b\cdot\vec{y}$ $a,b\in\mathbb{R}$ is a linear combination of $\vec{x}$ and $\vec{y}$. All vectors on the plane containing $\vec{x}$ and $\vec{y}$ are linear combinations of $\vec{x}$ and $\vec{y}$.
A2 Inappropriate Definition	A2a Good Attempt	S16	$\vec{v_1} = r\vec{v_2} + s\vec{v_3}$ ·addition of two vectors and their „Vielfaches"
	A2b Vague Attempt	S6	-when both vektors are having the same direction (parallelism)
		S12	zwei Vektoren z.B. in eine Ebene v_1 und v_2 v_1 und $v_2 \neq 0$ v_1 und $v_2 \neq 1$ v_1 und v_2 sind lineare-Kombination von einander wenn $v_1 = k \cdot v_2$ Translation of S12: Two vectors for example on a plane v_1 and v_2 v_1 and $v_2 \neq 0$ v_1 and $v_2 \neq 1$ v_1 and v_2 are linear combination one from another when $v_1 = k \cdot v_2$
		S9	-if you add one vector to another 
		S14	- Addition und Subtraktion von einzelnen Vektoren

Table 5.3: Students' Answers to the Question 2 in the Introductory Survey

property (Axiom 3a), of both the dot product and the determinants. Regarding the geometric mode of description, only one student, the exact student who provided a sketch of his/her image of the vector concept (S9) in Question 1, tried to provide a geometric visualization of this concept too, which shows that this student uses geometric modes at this beginning stage (S9 in Table 5.3). The student S11, who provided both geometric and arithmetic-algebraic mode of description of the vector concept in question 1, did not answer this question.

Regarding the *ARQ1* (p. 62), the overall analysis of students' answers to the both questions in the Introductory Survey suggests that the most of the students possess concept definition of *vectors* as classes of arrows and two of them possess a concept definition of vectors as ordered n-tuples in 3D. This means that their concept images primary base on the geometric mode of descriptions and thinking. The strong emphasis on this mode in students' concept definitions and concept images serve as a stable input for further development of the concepts in concern, the dot product (Section 5.2) and determinants (Section 5.3).

5.2 Analysis and Results Regarding ARQ2

The topic about the dot product has been part of the lectures in the course about two months prior this experiment was carried out. Students have been first introduced with a definition of dot product of two vectors in its component form, and then with the geometric definition utilizing cosine of the angle between the two vectors. This fact has been detected through the discussion with the teacher and was verified on the beginning of this teaching unit when definitions were revised in order students to refresh their previous knowledge (see Subsection 5.2.1). This situation suggests following the HLT (from Section 4.2) but preconditions its adaptation in further usage (or designs), according to which introduction of the concept may start with a geometric mode of description, towards component form definition ending with axiomatic-structural definition (all given in the corresponding Example 2 for dot product of vectors in Section 2.3, p. 44). While students were confident with the definition in the arithmetic-algebraic mode of description, they showed uncertainty with the geometric interpretation[1]. For this reason the Applet 1, p. 75 aims to contribute in widening students' concept definitions and concept images for dot product of two vectors by connecting it to oriented areas of plane geometric figures, rectangles and squares, preciously using vectors' projections in a DGE. Furthermore, the suggested Applet 1 supports the homogeneity Axiom 1a (Figure 4.4) in Section 4.2) and Axiom 3. Positivity (Figure 4.7 in Section 4.2). Task 4 (in the same Section 4.2) in a paper-pencil environment contributes to visualization of the symmetric Axiom 2 for the dot product.

5.2.1 Analysis Regarding the Algebraic and Geometric Modes of Description for the Dot Product

First, students have to explore Applet 1 and then try to interpret the (positive, negative or zero) value of the dot product of any two vectors depending on acute,

[1] This was also identified as a research problem (ii) through the mathematics forum in Subsection 3.1.1, p. 50.

obtuse and right angles spread between the vectors, in terms of elementary geometric concepts already known to them. By dragging modalities (see VDS, p. 82), changing lengths and directions of vectors and also angles between them they have to discover that dot product can be referred only to particular quadrilaterals, namely rectangles and squares and not other parallelograms. Furthermore, they have to convince themselves that the absolute value of the dot product equals the area of the obtained rectangle (one side of the rectangle equals the length of one of the vectors and the other side equals the length of the projection of the other vector) by the previously learned definitions. Possible students' misconceptions regarding the term oriented area are prevented such that positive areas are displaced on the rectangles while $\pm$ values of the dot product appear in the arithmetic-algebraic mode of description on the top of the Applet 1.
The teaching starts with a whole-class discussion about dot product definition, while students can see the applet on their monitors and simultaneously on the projector in the classroom. In continuation, here is the transcript of recorded actual teaching/ learning sequence for the dot product of vectors.

[1] Instructor: Do you know what a dot product of two vectors is?

[2] A couple of students: Yes [aloud in one voice].

[3] S1: So, it's a number!

[4] Students: [all laugh]

[5] Instructor: OK, so at least you know it's a number and not a vector. Which number exactly? How can we obtain this number? Maybe anyone could write it on the board?

[6] S2: [unclear vague explanation in German language, then writes definition of dot product in component form on the board, Figure 5.1].

[7] Instructor: OK, It's correct. Do you maybe know another definition of dot product of vectors? [addressing the question to the whole class].

[8] S3: [first writes cosine of the angle and then definition of dot product on the board without any comments, Figure 5.2].

Lines [1] to [5] in the excerpt from the transcript show that students already know that dot product of vectors is scalar (S1 in line [3]) and not a vector, which meets the first guiding feature towards conceptual understanding which is a distinction between what is and what is not a dot product of vectors (Subsection 2.1.1, p. 37). Further on, students by themselves experience (S2 in line [6], Figure 5.1 and S3 in line [8], Figure 5.2) that a concept may be given with more than one definition,

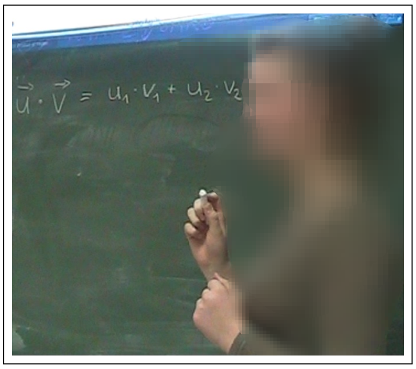


Figure 5.1: [6] S2 Writing Component Form Definition of Dot Product on the Board

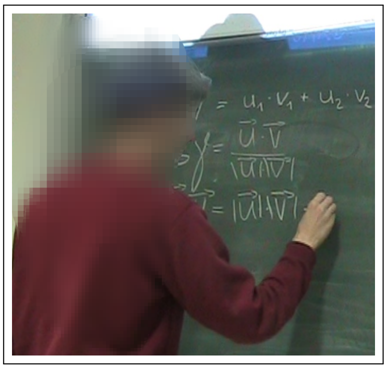


Figure 5.2: [8] S3 Writing Geometric Definition of Dot Product on the Board

which is the second guiding feature towards conceptual understanding (Subsection 2.1.1, p. 37 in this thesis). Student's S2 writing on the board (Figure 5.1) is actually the arithmetic-algebraic mode of description, while student's S3 writing (Figure 5.2) is the geometric mode of description of dot product of vectors. Further on, more formulas may exist even for a single mode of thinking of a concept, for example the formulas $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\varphi$ and $\vec{u} \cdot \vec{v} = \pm\,|\vec{u}|\,|\vec{v}_{\vec{u}}|$, both referring to the synthetic geometric mode of description and thinking of the concept. They are, of course, equivalent, though it may not appear to the students on the first look. Noticing and understanding this equivalence is one of the 'tasks' of the Applet 1.

What seem to be lacking is an oral explanation and understanding of the existing connections between the two definitions for the dot product. Namely, the student 3 first wrote the cosine of the angle and only after a request wrote the definition of the dot product of the vectors. This spontaneous student's reaction arrives from his previous exposure on the application of the dot product for measuring angles, thus confirms previous assumptions that the introduction of this concept in school is often limited to its application for measuring angles. Further on, students are able to write the geometric definition, but do not really understand what does it exactly mean. It seems that students are able to memorize these formulas, but their underlying understanding is symptomatic. Wittmann's case study about this concept shows the importance of memorizing formulas for solving tasks and exam problems for an interviewed student ([Wittmann, G., 2003], pp. 220-221). Memorizing formulas is certainly not all that we want to teach our students. Can technology facilitate situations like this one and how? One way to find out is through asking students for explanations in their own words when they use technological aids. The following excerpt of the transcript refers to **Applet 1. Dot Product** (Figure 4.1, p. 75).

[9] Instructor: [...] Could you explain what do you see on this applet? [Applet 1, p. 75]

[10] S4: So, we have the AF with the same length as AC, because

C and F are both on the circle around A. So we ... [turning to a student] Was ist Rechtecke? [asking for the word "rectangle" in English].

[11] Instructor: Rectangle.

[12] S4: Rectangle $AFED$ with two sides AC, so the length of the vector $\overrightarrow{AC}$ times the part of the vector $\overrightarrow{AB}$ which is ... ähm ... yeah, it's hard to explain [noticeable problem with his explanation into English language, lacking the word "projection"] which would shine directly onto $\overrightarrow{AC}$ [pointing on the screen] then the point D would be the shadow of the point B (Figure 5.3).

[13] Instructor: And how is that related to the definition?

[14] S4: Yeah, of course, here we have 90° [looking at Applet 1] so the cosine of the angle is this divided by this [pointing on the applet].

[15] Instructor: What do you mean "this dived by this"?

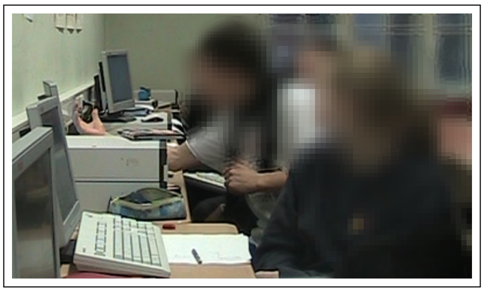


Figure 5.3: [12] Student 4 Explains the Applet for Dot Product

[16] S4: [laughing] of course yes. AD divided by AB ... is cosine of this angle between the vectors.

[17] Instructor: So in this way you find one of the sides of the rectangle and finally, what can you say about the area of the rectangle?

[18] S4: The area is the length of $\overrightarrow{AC}$ times the length of the $\overrightarrow{AB}$ times cosine of the angle, so therefore it is Skalarprodukt. [Skalarprodukt is the German word for dot product].

[19] Instructor: Thank you!

On the first instructor's question [9] for an explanation of the Applet 1, although without explicit use of the words "vector projection", i.e. correct terminology, the student S4 immediately recognized the projection and explained it in his own words with the phrases "[. . .] would shine directly onto [. . .]" and "the point D would be the shadow of the point B" in [12]. This student's discovery shows that he undergoes pure recognition because he is able to verify his understanding of concrete mathematical content articulating it using natural language. Besides performing the

VDS, the student additionally used his hands to explain his eureka on the computer screen (Figure 5.36). Such students' actions are described in literature as an embodied world of mathematics ([Tall, 2003]). On the next instructor's request ([13]) for justification of this student's explanation based on definition, the student successfully recalls the geometric definition and derives appropriate argumentation ([14] and [16]). A request for establishing connection between the dot product and the area of the rectangle by the instructor which followed ([17]) did not seem to confuse the student at all. He provided right away correct answer ([18]). The student argued that the area of the rectangle (with sides $|\vec{u}|$ and $|\vec{v}_{\vec{u}}|$ obtained by projection ([10] and [12]) corresponds to dot product of the vectors ([18], "... so therefore it is a Skalarprodukt")[2].

In relation to *ARQ2*, (p. 62) the whole above excerpt ([1]-[19]) shows how the student connected the arithmetic and the geometric mode of description (utilizing cosine of an angle and vector projections) for dot product of vectors with the help of the Applet 1. It seems that the DGE contributed in establishing interconnections between the two mentioned concept definitions, which directly meets guiding features 2 and 3 towards conceptual understanding (Subsection 2.1.1, p. 37 of this thesis). The analysis on the same ARQ2 regarding the axiomatic property for symmetry of the dot product continues in the next Subsection.

5.2.2 Analysis Regarding the Axiomatic Mode of Description of the Dot Product

Task 1. Dot Product (p. 80) for the Axiom 2. Symmetry of the Dot Product, is in relation to the fourth guiding feature (p. 37) about axiomatic properties of the dot product and aims to bring students closer to the formal-axiomatic world ([Tall, 2003]; 2004). In order to solve the task and learn the symmetric property, students have to transfer their applet-based experiences and undertake geometrical constructions in a paper-pencil environment. Thus, besides areas of plane geometric figures this task includes geometrical constructions of plane figures. This means that as already known entities for the students, plane geometric figures serve as inputs for learning the symmetric property of dot product of vectors, which is the Harel's concreteness principle ([Harel, 2000]).

Students either have to calculate dot product of the given vectors $\vec{u}$ and $\vec{v}$ using component form definitions and applying algebraic modes or have to interpret it geometrically as (plus or minus) area of the shown rectangle (which practically meets students' difficulty (ii), p. 50). The primary idea of the Task 4 is obtaining another rectangle (square) with the same area using the given vectors $\vec{u}$ and $\vec{v}$, only

[2] This is in connection to the question of the pre-service teacher posed on the mathematical forum [1] (in Subsection 3.1.1), on what does the resulting scalar represent exactly (see students' difficulty (ii), p. 50 and Research Problem, 61.

applying the other projection (of the vector $\vec{v}$ over $\vec{u}$). Yet, choosing new vectors (one or both) which fulfill the requirements in the task can only be accepted as *partly correct* solutions, because in such cases students successfully construct the dot product geometrically, but are probably not aware of its symmetric property. This would practically mean that students can successfully deal with guiding feature 3, p. 37, but maybe not with guiding feature 4, p. 37 of conceptual understanding. Categorization of students' answers is the same as in the Pre-study. Practically, there are two students who offered successful construction with the given vectors and applied the other projection (of the vector $\vec{v}$ over $\vec{u}$), thus provided an *appropriate* solution (Figure 5.4 and Figure 5.5).

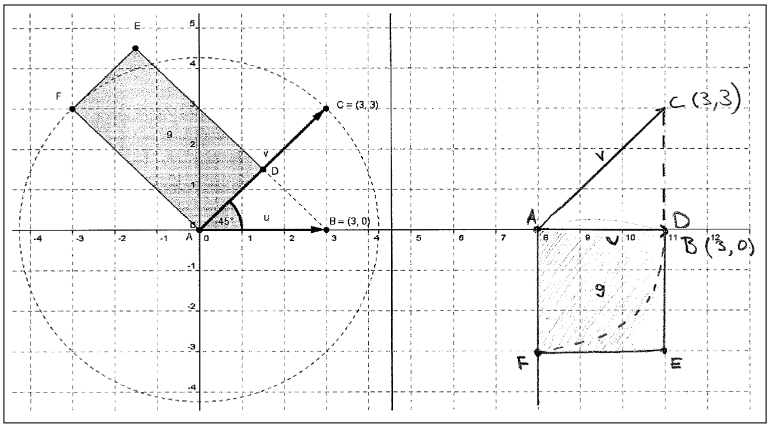


Figure 5.4: Geometric Mode of Description of Dot Product of Student 1

By applying this projection, both of these students draw a square with side 3 units and equal area as the given rectangle. These shows that they understand that, no matter which projection of the vectors is undertaken, it always leads to an area of a rectangle (or even a square as a special rectangle, in some cases) matching the $\pm$ dot product of the given vectors. That is exactly what symmetric property (Axiom 2) of dot product is about:

$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$, thus $|\vec{u}|\,(\pm\,|\vec{v}_{\vec{u}}|) = |\vec{v}|\,(\pm\,|\vec{u}_{\vec{v}}|)$

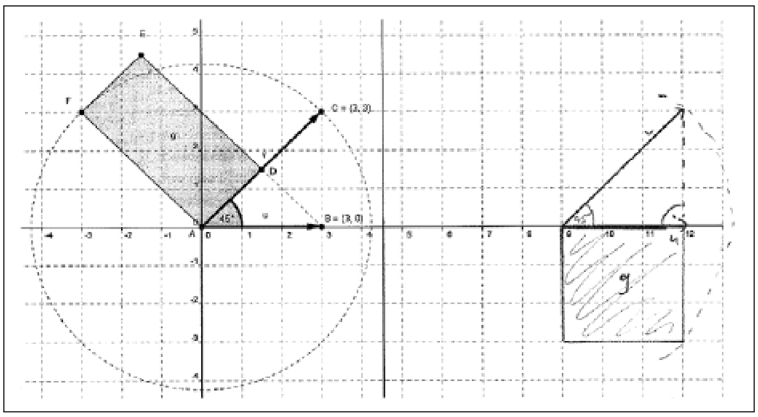


Figure 5.5: Geometric Mode of Description of Dot Product of Student 2

Another student (Figure 5.6) offered *partly correct* solution, because (s)he obtained a rectangle by keeping one of the given vectors same $\vec{u} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and choosing the other vector $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, which gives dot product equal to 9 (area of 9), but awareness of the symmetric property is discussable.

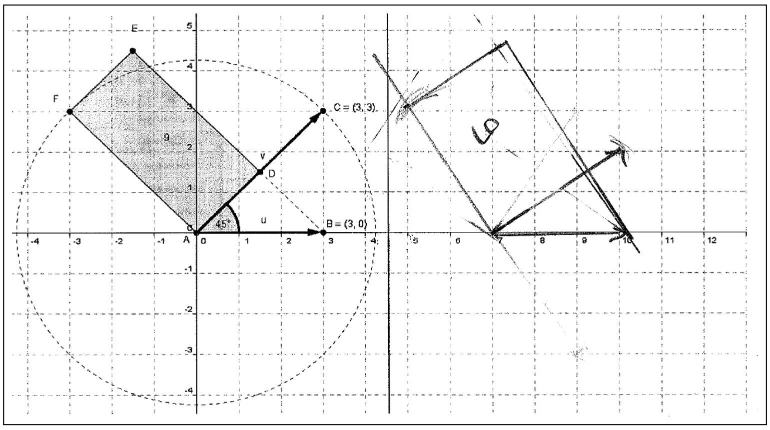


Figure 5.6: Geometric Mode of Description of Dot Product of Student 3

Similar as the previous student, student 4 keeps one of the vector same, in this case also the vector $\vec{u} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and chooses another vector $\vec{v} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ (reflection of $\vec{v}$ with respect to the x-axis) to construct dot product (Figure 5.7). Thus, in this case, the new vector has equal magnitude as $\vec{v}$ and $\cos(-\varphi) = \cos\varphi$. Although the vector projection and the geometric mode of dot product and its displaced value are correct, this solution does not offer another different rectangle as it is required in the task, so it is also partly correct (understanding of the symmetry is still discussable).

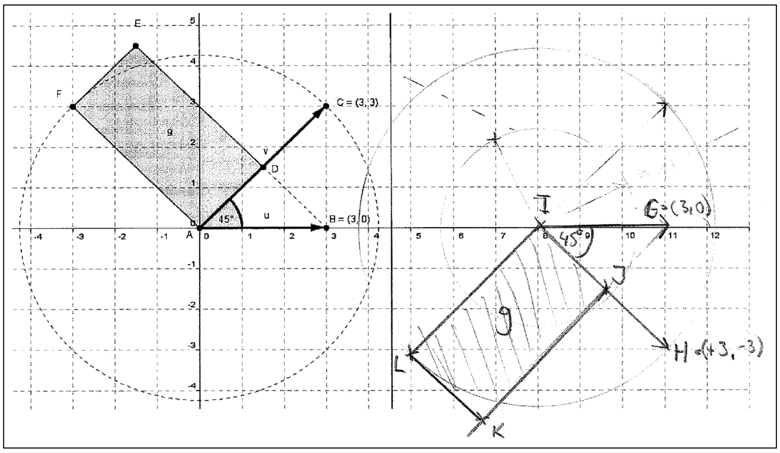


Figure 5.7: Geometric Mode of Description of Dot Product of Student 4

The next three students provided *good attempts* for the solution of this problem. Student 5 keeps only one of the given vectors same, namely $\vec{v}$, and chooses another vector different from $\vec{u}$ in both length and direction (Figure 5.8). (S)he understands vector projection and is able to construct the geometric mode of description of the dot product. The student also denoted that the area of this rectangle equals 3 squared units, which matches the dot product of the chosen vectors by the student. This shows that this particular student successfully translated between the algebraic and the geometric mode of description (probably calculating the value by the arithmetic formula). However, the required value is 9, so this solution is not classified as completely correct, but a good attempt.

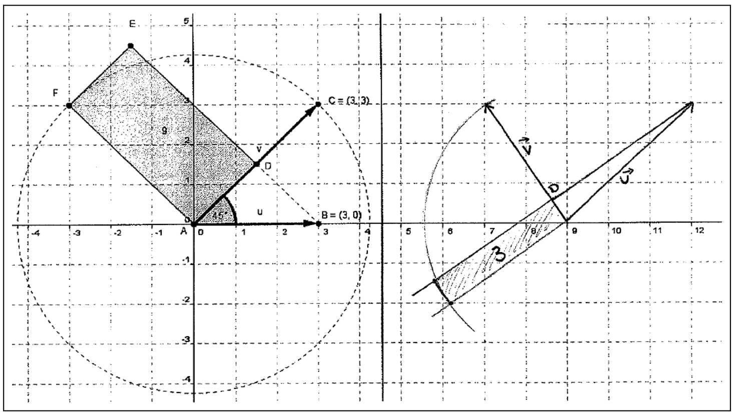


Figure 5.8: Geometric Mode of Description of Dot Product of Student 5

Student 6 chooses two completely new vectors and correctly constructs the dot product of these new vectors (by projection of one vector over the other one), but compared to the previous two students, the obtained rectangle is not highlighted (so it is not certain that the student can refer dot product to a rectangle and not to other quadrilateral) and there is no information about its area (Figure 5.9). It is also unclear whether this student can calculate the value of the dot product algebraically (equal to 17 according to student's new vectors $\vec{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ on student's drawing) and establish a connection between different modes of description.

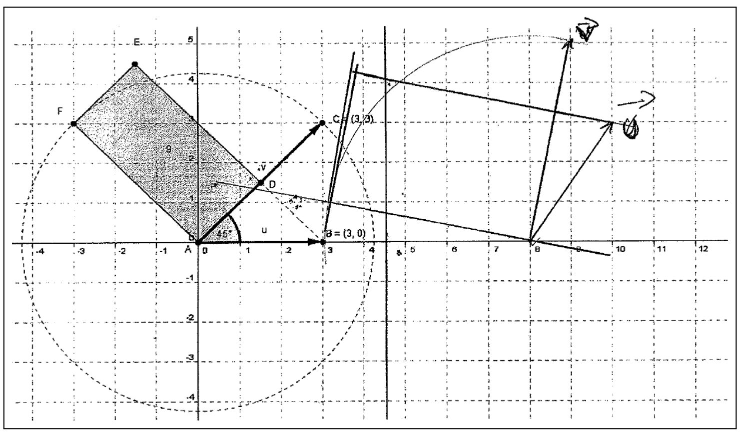


Figure 5.9: Geometric Mode of Description of Dot Product of Student 6

Compared to all other previous students' solutions, there is one which shows better capabilities for use of arithmetic-algebraic mode than geometric mode of description. Namely, notes on the top right corner of student's worksheet (Figure 5.10) illustrate that this student applied the component form of dot product of two vectors (s)he has chosen on her/his own, $\vec{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and correctly calculated their dot product. The problem which appears in the geometric representation is due to incorrect placement of the vector $\vec{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ in the coordinate system. Nevertheless, it is accepted as a good attempt because the student geometrically constructed dot product correctly.

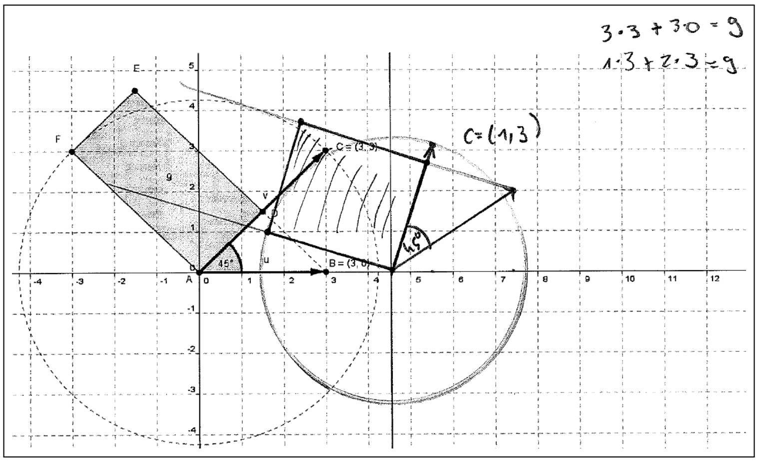


Figure 5.10: Geometric Mode of Description of Dot Product of Student 7

For the rest of the classmates, this task showed to be more difficult.
The next drawing (Figure 5.11) only shows that the student understands that a vector is a class of arrows equal in length and direction (which was part of the discussion on Question 1, p. 74, in the Pre-study, Subsection 5.1.1), but it is not certain whether the student understands geometric interpretation of dot product of vectors. For this reason it is classified as a *vague attempt.*

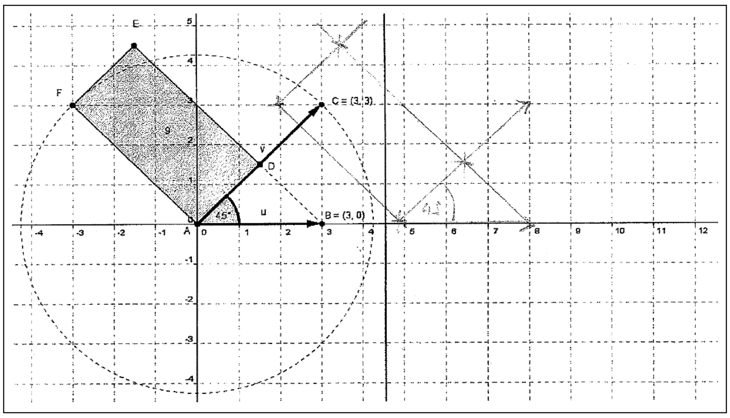


Figure 5.11: Geometric Mode of Description of Dot Product of Student 8

An *inadequate* student's solution is offered by student 9 (Figure 5.12) in which neither vectors are presented nor any construction, so it cannot possibly be discussed if the student is able to interpret dot product geometrically or translate between different modes of description.

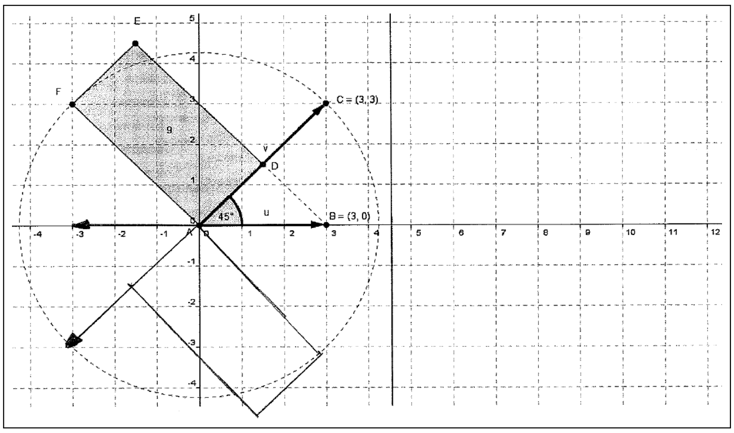


Figure 5.12: Geometric Mode of Description of Dot Product of Student 9

In summary, it was detected that despite students' rich concept images of vectors with strong emphasis on the geometric mode of description (according to the Pre-study, Section 5.1), prior the experiment took place, students possessed only limited visual representation of dot product of vectors. It is the mental construction of the oriented area of the rectangle which explains the meaning of the resulting scalar. In this experiment utilizing the Applet 1, through connections with projections of vectors and areas of rectangles, students were trained how to implement previously known formal concept definitions of dot product of vectors (Figure 5.1 and Figure 5.2) and widen their concept images. Namely, on the basis of the excerpt of the transcript of the video and students' written works, it seems that students integrated the arithmetic-algebraic and the geometric mode of description with the analytic-structural mode of description within the designed DGE, taking into consideration the embedded axioms into the design of the whole instrument (Applet 1, VDS, p. 82 and Task 1, p. 80).

5.2.3 Assessment for the Dot Product

Assessment through the Authentic Performance Tasks

Looking at the students' performance on the authentic tasks for the dot product, it seems that *oral communication* (the fourth data source and data set, p. 66) was an essential part of the learning processes. Students had to explain their thinking in any of the three modes of thought and justify their answers on the given questions and tasks. Their oral communication allows assessment of the degree of their conceptual understanding ([Hirschfeld-Cotton, 2008]). Here is an example.

The oral communication between the instructor and the student (lines [9] to [19], p. 106) when the student connects two concept definitions for dot product of vec-

tors (one, using cosine of the angle between the vectors, previously mastered by the student and two, a 'new' one, using vectors' projections on the applet) shows a successful student's externalization of personal perceptions and conclusions on the problem ([Kulikowich & Young, 1991]). This student's ability to articulate and distribute gained knowledge among the peers while working in the DGE is what distinguishes her/his achievement from the rest and places it at the highest level of understanding.
The categorization of the students' *written outputs* on the **Task 1** (p. 80) to appropriate solution, partly correct, good attempt, vague attempt and inadequate solution allows a distinction between the level of students' understanding of the axiom for the symmetry of the dot product. The presence of students' solutions in all of the categories shows that the DGE stimulated active participation of all students regardless the level of their achievements in the usual traditional instruction.

Assessment through the Mathematical Journals

When writing the Mathematical journals students used their own discretion by signing them with individual codes instead of their names. The writings in the journals were kept anonymous. This tactic secured students' confidence and openness in honestly sharing their personal assessment of understanding. The Mathematical journals were as well not graded, because the aim was providing feedback and communication between the students and the instructor and also supporting trustworthiness between them.
Prior the experiments students in the class were not familiar with this strategy and found it unusual ([Rosenstein et al., 1996], p. 604) and might have seemed at first surprising ([Borasi & Rose, 1989]), but after instructor's encouragement and explanation about the benefits from the journals, students collaborated to a greater extent. Here are some citations of their writings on two items in the journals for the dot product.
I understand:

- what a dot product is
- how to calculate a dot product algebraically (in 2 different ways) and it's absolute value
- how determine a dot product
- dot product of vectors, linear combination
- how to build the product of two vectors
- most of it
- most of the lesson

- very much, because I understand some things, I don't understand the last lesson

Students' reflections show that some them can already distinguish between computing and understanding of the dot product as one of them said "what a dot product is". Calculating in "2 different ways" as another students have said probably refers to the component form of the dot product of two vectors and the formula utilizing the product of the magnitudes of two vectors and cosine of the angle between them, thus, the arithmetic-algebraic and one geometric mode of description of dot product of two vectors. Furthermore, it seems that this students is aware of the geometric interpretation of the dot product of two vectors (adding "its absolute value geometrically"), which means that the student is probably aware of the meaning of oriented area. It seems that this student has obtained a wide conceptual understanding by multiple modes of description and visualizations of the dot product of vectors.
I do not understand:

- how to add vectors without use of a geometric graphic
- vectors in 3D
- linear combination of vector columns in the third dimension

All three students' writings directly refer to vectors and not to the dot product. The first and the third one show that the students do not understand the component form of vectors. This is a logic confirmation of the fact that the majority of the students had no prosperous background knowledge on the component form of vectors compared to the dominance of geometric modes of description and thinking about vectors, which was discovered through the questions in the Introductory Survey (see Subsection 5.1.1).

As a conclusion of the the analysis in this whole Section 5.2 regarding *ARQ2* (p. 62) it may be said the following. The described ALT shows how students were guided through the five features (p. 37):

1. what is and what is not a dot product of vectors (lines [1] to [5], p. 105),
2. reference to multiple concept's definitions (lines [5] to [19]),
3. modes of description (lines [5] to [8] and students' written works),
4. concept's properties which construct the axiomatic definition, in this case the property for symmetry (students' written works),
5. connections to areas of rectangles (students' written works),

all in contribution towards deeper conceptual understanding.
The most of the students performed not only procedural skills for calculating the resulting scalar of dot product, but moreover, they were required wider and deeper

thinking in order to compare the both definitions which they wrote on the board, discover and externalize their arguments for their findings with the Applet. 1 and solve Task 1.

5.3 Analysis and Results Regarding ARQ3

This part of the thesis focuses on investigations about multi-linearity property of determinants and their geometric interpretation[3]. These investigations are in close relation to the guiding features of conceptual understanding 2, 3 and 4 (in Subsection 2.1.1, p. 37). These investigations are undertaken in the designed DGE according to the discussing questions suggested in the hypothetical learning trajectory (HLT) in Section 4.3. Prior these investigations basic definitions and both algebraic and geometric modes of description of vectors are discussed. Particular attention is given to the component form of vectors due to the discoveries from the Pre-study which showed insufficient arithmetic-algebraic modes of description of vectors (as previously elaborated in Subsection 5.1.1). Operations with vectors (addition, subtraction, scalar multiplication and the dot product) and their properties (commutative, associative and distributive) are also part of this discussion. The didactical situation utilizing technology does not start with stating the axiomatic definition of determinants, but rather with exposing students on explorations of the homogeneity property (Axiom 3a in Table 4.1, p. 84) with the **Applet 1. Determinants** (Figure 4.8, p. 83). One of the reasons for not starting with the Axiom 1, so starting with a parallelogram and not with the unit square, is that research does not favour special cases in teaching Linear algebra. It leads to prevention of students' hurdles into the generalization from a lower to a higher dimension as a natural process and into the transition from the arithmetic to the algebraic language and the corresponding arithmetic-algebraic mode of thinking to the analytic-structural mode of thinking of the general theory, as defined by ([Hillel, 1997]) and ([Sierpinska, et al., 1997]). So, on the beginning, the instructor describes Applet 1. Determinants. Students are already familiar with the GeoGebra interface, so the focus is primary set on two different modes of description of vectors and determinants represented by matching colours (red and blue) on the Applet 1. How students explore that a determinant is linear in each column (row) by the aid of sliders (Applet 1) and how they connect areas of plane geometric figures with vectors and determinants, can be seen from their verbal communication and collective instrumental genesis in the following excerpts of the recorded lecture.

5.3.1 Analysis Regarding Axioms 2 and 3a for Determinants

The Actual Learning Trajectory (ALT) referring **Applet 1** for **Axiom 3a** and **Axiom 2** following discussing questions from the Hypothetical Learning Trajectory

[3]These are students' difficulties (i') and (ii') identified in Subsection 3.1.3, p. 61.

(HLT) (in Section 4.3) is as follows.
[...] 33:52

[1] Instructor: Double the length of one side of the parallelogram $ABCD$. [...] Set one of the sliders at 1 and the other one at 2. [...] How does this change affect the area of the parallelogram? What do we get here? How can we find the area? How can we read from the applet? Is there any information given about the area?

[2] S1: The area of the new parallelogram is now twice as big as the one given.

[3] Instructor: Yes, correct! Now you have to write it down, thus answer the question 1a [addresses to all students, then explains matching colors of determinants and displaced areas]. Pay attention to the entries of the determinant and the coordinates of the points, vertices of parallelograms. Is there any connection? Change different positions of the vectors by moving the ending points of the vectors $\vec{u}$ and $\vec{v}$ and notice how does it affect the entries of the determinant. Do you understand?

[4] S2: I don't understand.

[5] Instructor: You don't understand the question. OK, I will repeat and explain once again the question. How is the area of the parallelogram related with the determinant? So, you move the ending points of the vectors, for example point B or point D and you can also change the sliders and you have to find how this area of the parallelogram is related to the corresponding determinant.

[6] S2: Should I write the vectors down and then the area?

[7] Instructor: For example, there are some numbers written here [pointing on the projected screen]. Which are those numbers? How are they chosen? [...] How are they written in the determinant?

[8] S2: Yeah [...] First is the $\vec{u}$ vector, that is up and the second vector is down [pointing horizontally on the determinant on the Applet 1].

Students' first exposure to the dynamic geometry environment requires some time for them to adopt and moreover instructor's patience and effort to support them in situations (as [3]-[5]) which are not easy, but motivating for them. The student S2 (line [4]) was not afraid, but somewhat stimulated by instructor's question in [3], to admit that (s)he does not understand what is required. It seems that a relation of trust among the students and the instructor has previously been established. This is a decisive factor for students not to give up, but continue working on the theme with enthusiasm. Thus, after additional explanation of what the question requires, student S2 (in line [8]), although using informal mathematical language saying that "the $\vec{u}$ vector, that is up and the second vector is down", successfully discovers how are vector components displaced as entries in each row of the corresponding determinant. Awareness of the relation vector $\rightarrow$ determinant's row (column) is a necessary precondition in understanding multi-linearity (in this case bi-linearity) of the determinant function, so the discussion continues in this direction.

[9] Instructor: So the entries of the first vector $\vec{u}$ are written as the first row, 3 and 1, and then, the other vector with coordinates 1 and 2, these are the entries of the second row, the coordinates of the second vector $\vec{v}$. [...] On the worksheet you have to write answer of the question 1b. Those who are finished could go on with the question 1c. Compare the length and the direction of the vectors $\vec{u}$ and $e\vec{u}$ (or $\vec{v}$ and $k\vec{v}$). What is their relation to the determinant? What do we notice for the vectors $\vec{u}$ and the vector which is obtained by multiplying $\vec{u}$ by a scalar e? e is a real number, as we can see here [pointing on the slider] represented by the slider in red color. What can we say about these two vectors? We mentioned this last time when we discussed linear combinations of vectors. What kind of vectors are these? [...]

[10] S3: They have the same direction, but $e\vec{u}$ is longer.

[11] Instructor: In this case, so as they are represented here [pointing on the applet] they have the same direction and the length is bigger [...], but we have to emphasize here that in our case e is positive. Then the vectors have same directions and the vector $e\vec{u}$ has bigger magnitude. But, in general you have to be very careful. What did we explain last time? Depending on e, we may also have something else here. What do you think? [addressing to student S3]

[12] S3: I'm not sure.

[13] Instructor: If e is a negative number? What kind of a vector do we obtain?

[14] S3: Negative vector.

[15] Instructor: Can a vector be negative?

[16] S4: Changes. Direction changes!

[17] Instructor: [Drawing two arbitrary vectors on the board] Can it be like this for example?

[18] S5: No.

[19] S6: Opposite [whispering from behind].

[20] S4: Goes the other way.

[21] Instructor: Yes. So, it has opposite direction. If the scalar is negative, the new vector has an opposite direction of the first one. What about the magnitude? What can we say about the magnitudes of the vectors $\vec{u}$ and $e\vec{u}$ if the scalar is a negative number?

[22] S1: For example for e, 1 and -1 vectors have same magnitude, but other direction. And like -5 is with bigger magnitude than -4 but, 1.5 would be smaller.

[23] Instructor: Yes, so actually it depends on the absolute value of the number e and if it is a number less than 1, then the new magnitude is smaller. If it is greater than 1, then the new vector has bigger magnitude than the given vector. So that's the explanations about the question 1c on the worksheet.

The situation [9] to [23] is a typical situation in which 'guessing' is a desirable students' activity and very much facilitates development of intuitive thinking ([Bruner, 1966], p. 64). Students' instinctive reactions as: "$e\vec{u}$ is longer" (S3 in [10]), non-considering that e may also be a negative number; "Negative vector" (S3 in [14]) or lack of precision as: "changes" and immediate addition of "Direction changes" (S4 in [16]) confirm that this revision of previously studied material about vectors and operations with vectors was a necessity. However, students are not in any case 'penalized' for not giving immediate correct answers, but encouraged to transparently share their previous knowledge by talking and current heuristics by posing new questions, as for example in [17] when the instructor offered simple geometric sketch of non-collinear vectors on the board. Such instructor"s attitude immediately resulted with correct answers by three students S5, S6 and S4 (in [18], [19] and [20], respectively). It seems that these spontaneous interactions between the instructor and the students contributed to sharing a successful reasoning about the magnitude of the vector by the student S1 (in [22]). Although this student does not explicitly use the term "absolute value", by stating "[...] -5 is with bigger magnitude than -4 but, 1.5 would be smaller", it seems that absolute value is what the student meant. The instructor than only summarizes the discussion using mathematics terminology. Inclusion of as many as possible students into the oral conversation is an important feature for students' motivation. Openness and self-confidence, even in situations when students do not know the answer ("I'm not sure" by S3 in [12]), lead to further students' curiosity (the same student S3 was not discouraged to ask new questions, as in [34] and arise with constructive conclusions as in [43], below in this text). This same fact can be also confirmed by following student's S2 communication, starting from "I don't understand" (in [4]) and finishing with [33] in the next excerpt.

Evoking students' concept definitions and concept images of vectors (magnitudes and directions) and this brief revision of the operation scalar multiplication through an oral communication, similarly as through the Pre-study in written works, was necessary intuitive introduction into teaching the multi-linearity property of determinants, in particular the Axiom 3a, as follows.

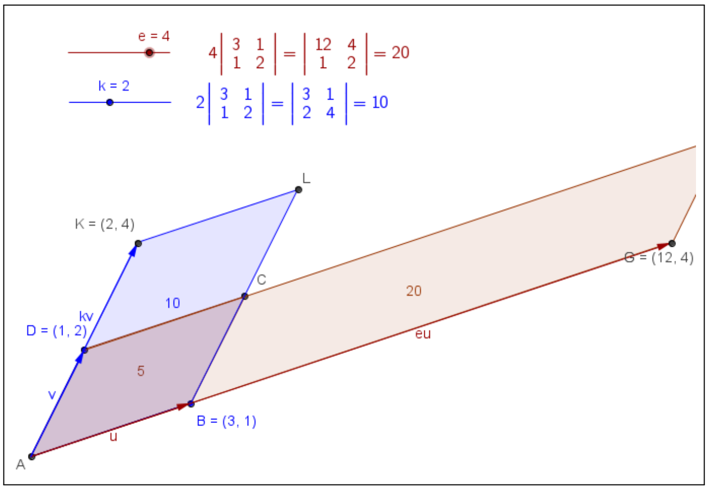


Figure 5.13: Position of Applet 1 for Discussion about the Homogeneity Axiom 3a in [24]

[. . .] 48:12

> [24] Instructor: Can we go on to the next questions? [. . .] What is the general thing, very important, that we have to notice? Look at the first determinant. We have 4 multiplied by a determinant [setting the slider to $e = 4$, as on Figure 5.1], with entries 3, 1 and 1, 2? [$4\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$] What do we have here?
>
> [25] S2: The same multiplication with the number.
>
> [26] Instructor: What is multiplied by the number?
>
> [27] S2: 4 with 3, 12 and 4 with 1, 4 [pointing on the projected screen]. But I don't understand why the one then don't be multiplied?
>
> [28] Instructor: That is a very important thing. Yes, can one of you explain?!
>
> [29] S4: Because the multiplication 4 with the one, so if you multiply the one also with 4, you have multiplied two times with 4.
>
> [30] Instructor: Exactly! It means that you have to multiply again by 4 here [pointing on the applet]. So, how is this operation defined? So we have operation multiplication of a scalar (or of a number) and a determinant? How do we multiply? We multiply exactly one of the rows, only one, either the first or the second, but not both at the same time. Because if we multiply both it will not correspond to this area.
>
> [31] S2: But sometimes there is anything else multiplied?!
>
> [32] Instructor: Do you mean when you move the other slider, or?

> [33] S2: Yes, sometimes there is in the first and up, is multiplied, and the second down [murmuring while moving the sliders on the applet]. Oh no, nothing. It's OK.

The fact (line [27]) that the question, why only one row in a determinant is multiplied by a scalar and not both rows, was posed by a student (S2) and not by the instructor confirms not only instructor's abstinence of influencing students' thoughts, but students' curiosity and reasoning which emerged in [1] to [8] is now growing in the DGE. This student searched for relevant argumentation of her/his own heuristics. The instructor resisted to answer the question immediately, instead emphasized the importance of the discovery and encouraged other students to explain it (line [28]). Student S4 instantly responds to this instructor's 'provocation' and reports on her/his finding that if both rows are multiplied by the scalar 4 then "you have multiplied two times with 4" (line[29]). Instructors' explanations about linearity in a single row now follow (line [30]). Meanwhile student S2 continues to explore with both sliders on the applet and finally (line [33]) realizes their separate affects in exactly one of the rows in each of the determinants, thus verifies the answer. This shows a relevant fact that the DGE may stimulate students' explorations and supported them in posing own questions.
Starting with area of a parallelogram and student's (S1 in line [2]) immediate detection that one slider affects the area by multiplying it with the corresponding number (slider), through student's hurdles in verifying it with vectors (S2 in lines [8] and [33]), till student's (S4 in line [29]) final conclusion, they connect concepts (areas of parallelograms, vectors and determinants) and modes of description (geometric and arithmetic-algebraic). Their collaboration in the whole-class discussion is meaningful. This whole above excerpt of the transcript ([1] - [33]) appears to be efficient in learning the muli-linearity (more preciously, homogeneity) of determinants[4]), when students meet these concept for the first time and they are still at the upper-secondary education. The conversation continues with a sudden question. Right after student's S2 self-confirmation in her/his discovery, the student S3 (in [34]) asks a question addressing diagonal entries and changes of determinants' values. This situation is used by the instructor as a smooth introduction to Axiom 2 by saying that certain changes of determinant's entries have some consequences (in [39]). Thus, raising awareness that existence of certain 'rules', namely Axiom 2, ("property number 2" in [41]) regulates these changes of determinants' entries and their positive or negative values (in [41]).
[51:13]

> [34] S3: But you cannot change, you cannot choose the same diagonals, right [pointing on the diagonal entries of determinants on the Applet]?

[4]The approach may also be used in similar situations in order to prevent students' difficulty (i'), which was identified in Subsection 3.1.3, p. 61

[35] Instructor: No.

[36] S3: Because it's a different . . .

[37] Instructor: Result?

[38] S3: Result.

[39] Instructor: No, we cannot just change or write the elements in any other order that we would like because that will affect something else.

[40] S3: OK.

[41] Instructor: It is written on the worksheet as property number 2, [. . .], and you can see it also on the applet. How does it affect the sign of the determinant? Property number 2, pay attention here [showing on the worksheet], [. . .] says: if you change the order of the two rows, so first you write the second row to be first and the first one to be second, then you will get a negative value of the determinant.

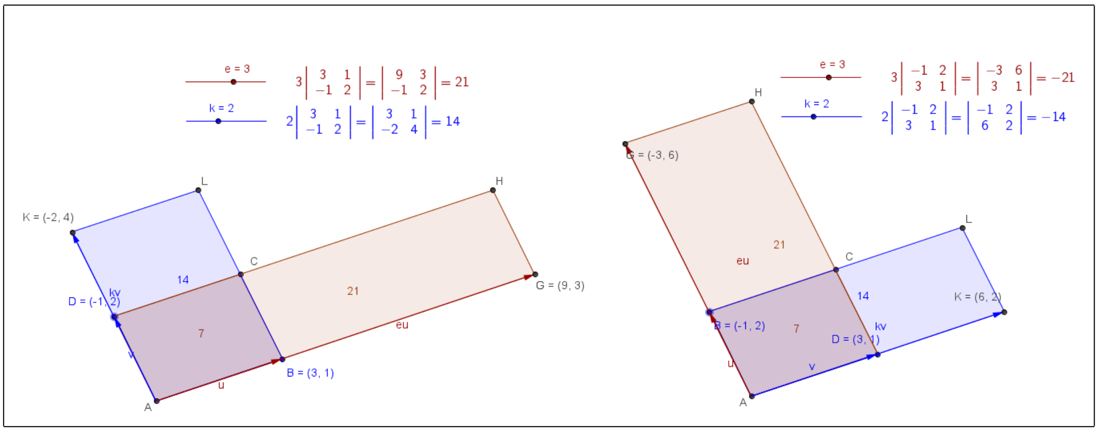


Figure 5.14: Dynamic Visual Demonstration of Axiom 2 with the Applet 1

Then the instructor sets the Applet 1 in positions as on the Figure 5.14 in order to demonstrate how counter-clockwise (Figure 5.14 left) or clockwise (Figure 5.14 right) changes affect the sign of the determinants. Then, the instructor writes the property: changing two rows (columns) in a determinant, changes its sign, (Axiom 2, p. 84) on the board and also shows it with the rule of Sarrus. During this demonstration sliders are kept unchanged in order prevention of confusions. With this dynamic visualization on the projected screen and instructor's writing on the board, discussion on the axiomatic property 2 with the aid of the same Applet 1 is completed according the planned HLT (in Section 4.3).

5.3.2 Analysis Regarding Axioms 2 and 3b for Determinants

The Actual Learning Trajectory (ALT) referring **Applet 2. Additive Property of Determinants** (Figure 4.9, p. 83) for **Axiom 3b** (Table 4.1, p. 84) following discussing questions from the Hypothetical Learning Trajectory (HLT) (in Section 4.3) continues as follows. First, the instructor provides instructions about the Applet 2 (Figure 5.15), pointing out the movable points B, C and D which are terminal points of the representatives of the vectors $\vec{u}$, $\vec{v}$ and $\vec{z}$, respectively. Then students investigate the Applet 2. The discussion starts with the fifth discussing question from the HLT (Section 4.3), what is the relation between the vectors $\vec{u}$, $\vec{v}$ and $\vec{z}$, and determinants, which is simultaneously the first question for this particular Axiom 3b and Applet 2 (line [42] in the next excerpt of the transcript).

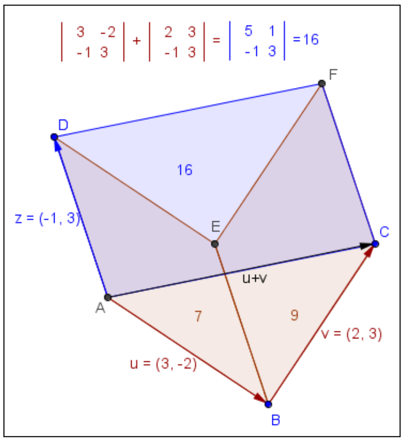


Figure 5.15: Initial Position of Applet 2 for Axiom 3b

[...] 1:01:30

> [42] Instructor: What is the first question? We have to find a relation between the vectors $\vec{u}$, $\vec{v}$ and $\vec{z}$, and the determinant? Who is going to answer the first question? What can we see here?
>
> [43] S3: [pointing on the Applet] The first row of the first determinant is $\vec{u}$, the first row in the second determinant is $\vec{v}$ and the second row of the first, second and third determinant is $\vec{z}$.
>
> [44] Instructor: Yes, that's correct. What do we have as a result of this addition of two determinants? [...] Let's explain them. [...]
>
> [45] Instructor: Entries in the first row are obtained by addition of the corresponding entries of the first rows and entries in the second row are the same. We do not add the entries in the second rows. They are not changed, they stay the same.

The above discussion continues to focus on the bi-linearity property of 2×2 determinants (additive property), i.e. Axiom 3b. The transcript shows that a student

(S3, line [43]) noticed that the second rows in all three determinants remained unchanged. Thus, additive property in each row, preserving the other row same, seems to be partly clear at this moment. All three parallelograms have one side equal in length (magnitude of $\vec{z}$), but what remains unclear is whether the students can autonomously also detect that the first row in the last determinant is a sum of the other first two, thus recognize vector addition in the algebraic mode of description. For this reason, after students' trials, the instructor took the role in the explanation (in [45]). It seems that this students' easier detection of geometric addition of vectors $\vec{u}$ and $\vec{v}$, but not so easy detection of its component form is a consequence of previous mainly geometric approach in teaching/ learning vectors confirmed with the Pre-study, 5.1.1). Line [49] in the next excerpt of the transcript illustrates how a student detects addition[5] of determinants in geometric mode of description. The same line ([49]) also shows that students are able to connect the geometric and the arithmetic mode of description of determinants presented on the Applet 2. Namely, S1 says: "Areas of parallelograms are the determinants, so the first two [...]", meaning the first two out of all three determinants shown on the top of the Applet 2 (Figure 5.3).
[1:05:45]

[46] Instructor: The second question asks, what is the relation between parallelograms and determinants? How many parallelograms do you see on the screen?

[47] S7: Ähm... [other students whispering Zwei, Drei] Three.

[48] Instructor: OK, and what are their areas? For example $ABED$?

[49] S1: [looking at the projected screen] Areas of parallelograms are the determinants, so the first two and the parallelogram $ADFC$ is sixteen [Figure 5.3].

[50] Instructor: Yes. And what do you think; can a determinant have negative value?

[51] S6: Yes. Yes, it can.

[52] Instructor: [...] if we have a negative value of a determinant, how is it related to the area of the corresponding parallelogram?

[53] S6: The area is the value of the determinant. Probably.

[54] Instructor: But what if the value is negative?

[55] S6: [moves points on the Applet] Ähm, man! [shows tense reaction and searches the short dictionary]. The absolute value!

[5] This is in connection with the investigations about the property $\det(A+B) = \det(A)+\det(B)$, which was discussed between university students and a teaching assistant, when multi-linearity was identified as students' difficulty (ii') in Subsection 3.1.3. p. 61.

[56] Instructor: Absolute value! Yes. [Setting points on the projected applet in such positions to illustrate negative values of determinants and their absolute values representing corresponding areas of parallelograms.] [...] That's why I say you have to experiment with the applet and set points in different positions.

Although the ± signs of determinants (Axiom 2) were discussed with the previous Applet 1, the instructor asks whether determinants can have negative values again (in line [50]). The aim for raising the same question is this time different, i.e to connect it with the geometric interpretation of determinants[6]. Namely, the aim is to show that their absolute values equal the areas of the parallelograms spanned by corresponding vectors. Students' accomplishments (S1 in line [49] and S6 in line [55]) regarding this issue seem evident. The excerpt of the transcript ([42] - [56]) shows a possible way of learning about the geometric meaning of determinants at this level of education. It may contribute in establishing a fundamental for developing of rich concept images which involve geometric modes of description (guiding features 2 and 3, p. 37).

5.3.3 Analysis Regarding Axiom 1 for Determinants

The next part of the discussion continues with focus on Axiom 1 (Table 4.1, p. 84 with the Applet 1 (Figure 5.16) projected on the screen. Focus on the Axiom 1 is important because students misconceive the determinant of the unit matrix and sometimes view it as the unit matrix itself ([Aygor & Ozdag, 2012]). Generalization of the Axiom 1 for determinants in order three, using analogy is as follows:

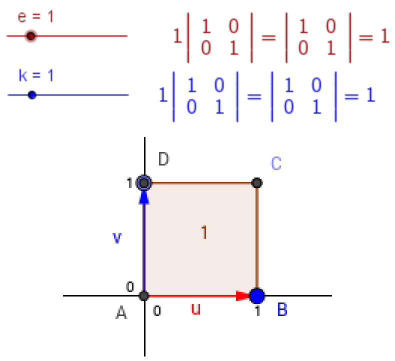


Figure 5.16: Visual Demonstration of Axiom 1 with the Applet 1

[...] 1:20:17

[57] Instructor: Let's focus now on the question number 3. We have

[6]The geometric interpretation of determinants was identified as students' difficulty (ii') in Subsection 3.1.3, p. 61

to set the points B at $(1, 0)$ and D at $(0, 1)$. What geometric figure is obtained and which one of the axioms is obtained?

[58] S: [sets the points B and D and the sliders in the required positions] Square.

[59] Instructor: It's a square, yes and we can see how the determinants match with the coordinates of the vertices of the new figure, with the square and the final determinant that we obtain with entries 1, 0, and 0, 1 is called identity and its value is 1. Which of the properties listed on the worksheet is that? [...]

[60] Instructor: What happens if we have determinants with three columns and three rows? How can we find the value of this determinant? [...] If determinant in order two is related with areas of a parallelogram, then determinant in order three is related to ... what?

[61] S4: Quader. [translation: Cuboids. Immediate reaction and association to solids, but with linguistic problems].

[62] Instructor: Quader or?

[63] S6: The area aber in drei D. [translation: Area, but in three D].

[64] Instructor: Not the area, but?

[65] S1: Volume.

[66] S6: Parallelogram in drei D. [translation: Parallelogram in three D].

[67] S4: Parallelepiped. [with uncertainty in the English pronunciation].

[68] Instructor: Yes, parallelepiped. Absolute value of the determinant represents the volume of parallelepiped in three dimensions. [Sets homework problems and Mathematical Journal for the next lecture].

All of the above excerpts of the transcripts confirm the need of instructor's support (due to the high cognitive demands and in this study due to the utilization of a foreign language of instruction) when students work in a computer-based interactive environment repeatedly suggested in literature. Namely, guiding questions are often proposed as a supporting measure ([Njoo & de Jong, 1993]; [Swaak et al., 1998]) in carrying out "instructions that do not overburden the learner's working memory capacities" ([Bodemer et al., 2004], p. 327). Instructor's guidelines aiming to assist and increase learning performances are of a particular value on the beginning of the undertaken experiments utilizing dynamic and interactive visualizations and they underline the step-by-step presentation of the conceptual model to be learned (eg. [Swaak et al., 1998]). Simultaneously, students: (1) focus on particular modes of description embedded in the Applet (in [51] student's S6 focus on determinant's value, thus arithmetic mode), (2) generate hypotheses about relationships between modes (S6 in [53]: "The area is the value of the determinant. Probably.") and (3) test and evaluate the hypothesis (S6 in [55]: "[...] The absolute value".) (according [Bodemer et al., 2004], p. 327).

The ALT is not yet finished by fulfilment of the class activities, but it continues with students' engagements on assignments as homework problems and Mathematical journals, as additional supporting measure for improvement of learning processes in DGE ([Njoo & de Jong, 1993]). The point of the Mathematical journals was discussed in details at the end of Subsection 3.3.2, p. 70, but here a particular task from the journals is pointed out. It aims to contribute in supporting conceptual understanding through analogizing and thinking in general terms, which is the third indicator for understanding according [Harel, 1997], p. 109. More preciously, students generalize Axiom 1 for determinants from two to three dimensions through a paper-pencil homework assignment after the whole class discussion [1] to [68]. So, in this task, first, students have to provide geometric mode of description for the Axiom 1 in three dimensions, thus to draw the unite cube in a three dimensional rectangular coordinate system. Then, guided by a small set of questions (HLT, Subsection 4.2.1) about a) the volume of the unit cube and b) the coordinates of its vertices, they have to translate this geometric mode into algebraic mode of description, i.e. to write the required determinant of the unit 3×3 matrix in the question c). Students' solutions of these tasks are categorized as in the Pre-study and now follow.

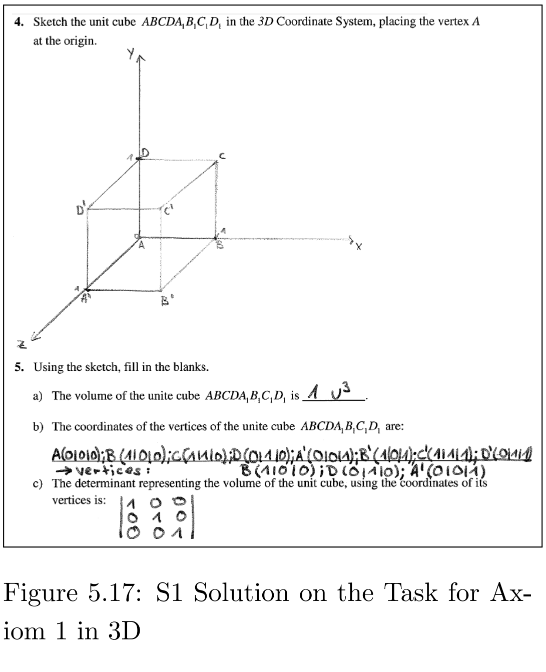


Figure 5.17: S1 Solution on the Task for Axiom 1 in 3D

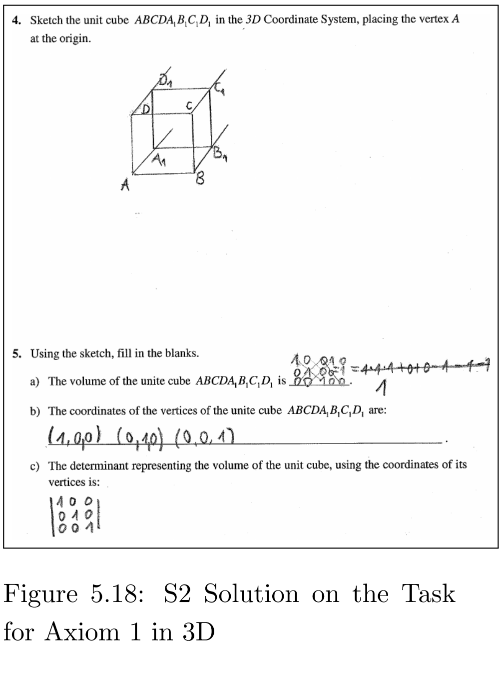


Figure 5.18: S2 Solution on the Task for Axiom 1 in 3D

Students' solutions S1 and S2 (Figure 5.17 and Figure 5.18) on the homework problem are categorized as *completely appropriate solutions*, because both the geometric and the algebraic mode of description for the determinant of the unit matrix in three dimensions are correct. In student's S2 solution an attempt for calculation of the determinant using Sarrus' rule is noticeable on the paper (Figure 5.18). The student obtained an incorrect result of this calculation, but probably realized this in-correctness, then tried to erase it (with a line over the calculation on the paper) and finally stated the correct result.

An *appropriate geometric mode of description*, but *incomplete algebraic mode of description* was detected in three other students solutions S3, S4 and S5 (Figure 5.19, Figure 5.20 and Figure 5.21).

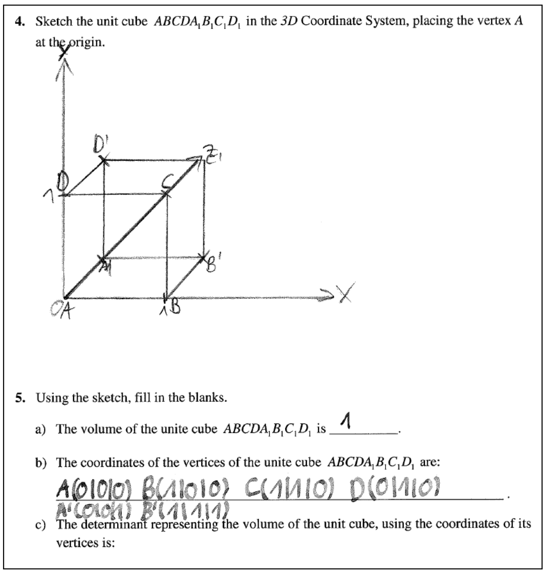


Figure 5.19: S3 Solution on the Task for Axiom 1 in 3D

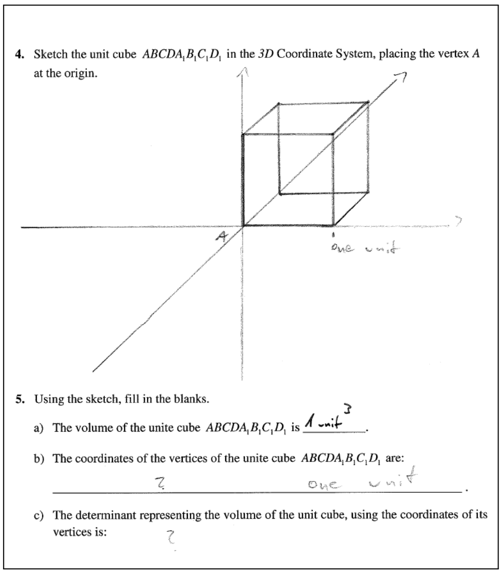


Figure 5.20: S4 Solution on the Task for Axiom 1 in 3D

Regarding the geometric mode of description, S3 and S4 provided correct final value for the volume of the cube, namely 1 cubed unit, but S5 remained captured by the geometric formula for the volume as a product of length, width and height ("$AA' \cdot AB \cdot AD$" on the student's paper in 5a), Figure 5.21) forgetting to find that product. On one hand, this student's writing may be viewed as impossibility to generalize Axiom 1 from two to three dimensions, but on the other hand, it may be interpreted as an advantage for generalizing the geometric mode of description for determinants representing (oriented) volumes of boxes (and then parallelepipeds) in coordinate-free geometry. Thus, this solution shows dominant students' abilities in synthetic geometry. Regarding the algebraic mode of description, students' solutions S3 and S4 differ. While S3 wrote the coordinates of the vertices of the unit cube in the requirement 5b) (Figure 5.19), S4 did not offer an answer (Figure 5.20). One reason for it may be the difficulty to transfer representations from synthetic to coordinate geometry. Also, none of these students S3 and S4 answered 5c).

These students' home-works were analysed by the instructor and brought into dis-

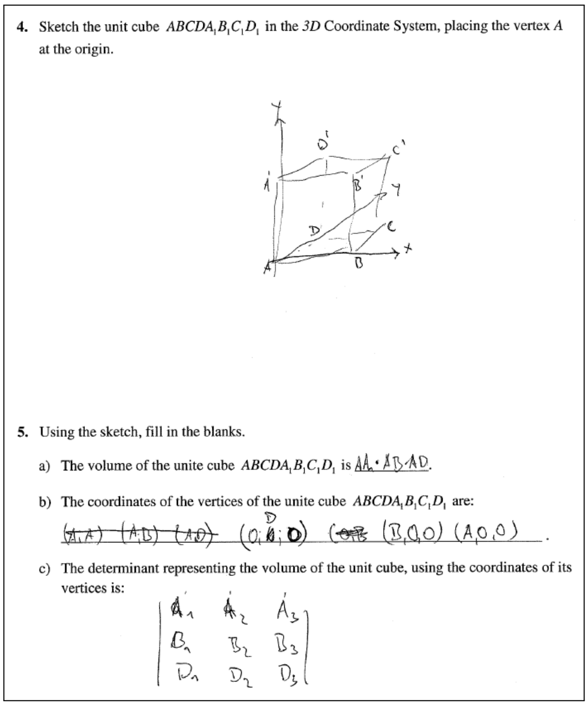


Figure 5.21: S5 Solution on the Task for Axiom 1 in 3D

cussion, together with the writings from the Mathematical Journal again on the beginning of the next lecture. A static visualization of Axiom 1 in 3D (Figure 4.11, p. 87) is then also presented to the students in order to facilitate this generalization from two to three dimensions using analogy.

A short conclusion of the above elaboration is the following.

Besides questions framed within the HLT (Section 4.3), the instructor also asks questions as: do you understand (line [3]), what do you think (lines [11] and [50]), can anyone explain (lines [28] and [44]). Such ad hoc decisions for posing these questions aim to stimulate students in sharing their own reasoning and thinking while observing and exploring, to promote transparency of students' heuristics, to support students in making their assumptions explicit so that others can respond and give feedback and to strengthen overall interactions in a technology-rich classroom. The whole above didactically situation in technology-rich arrangements has its counterpart in a paper-pencil environment and it is elaborated in Section 5.4).

The above discussion demonstrates how students communicate mathematical ideas in their own words, for example: S2 in [8] an idea for linearity in a row, S1 in [22] an idea for scalar multiplication of vectors, S4 in [29] and S2 in [33] an idea for the homogeneity Axiom 3a, S3 in [43] an idea also for linearity in a row, S1 in [49] an idea for the additive Axiom 3b, S6 in [53] and in [55] sense for the role of the sign, i.e. Axiom 2, S6 in [66] an idea for generalization in 3D. It is the second indicator for conceptual understanding according [Harel, 1997], p. 109, which shows how "natural language was used as a vehicle for describing iconic representations" ([Tall, 1995], p. 7). Thus, through physical actions (dragging sliders, moving points, changing vectors) or students' enactive exposure to applet-based learning (accomplishing the VDS) they achieved progress in cognition. Talking, even in incomplete sentences, enables students to establish connections between their previous knowledge and new ideas, which is the fourth indicator for conceptual understanding according to [Harel, 1997].

The above described ALT represents a mini-story on how students learn about determinants through connections with vectors and parallelograms in a DGE and translating between different modes of descriptions, as suggested in the HLT (Section 4.3). It shows that the designed HLT in the DGE may be used for preventing occurrence of possible obstacles (e.g. (i') and (ii') identified by the research problem in Subsection 3.1.3). Particularly, the above excerpts of the transcripts for each of the Axioms 1, 2, 3a and 3b testify this assumption. It promotes conceptual understanding through the guiding features 1, 2 and 3 and a concept definition for determinants and wide concept images formed on the bases on translations between different modes of description. Having the geometric and arithmetic-algebraic modes and language of descriptions embedded in an applet-based approach supporting axiomatic properties, the DGE enables the algebraic-structural mode of description and thinking in the general and unifying theory of Linear algebra. Elaborations of the part of the ALT referring geometric solids, vectors and determinants in order three appear to be an appropriate introduction to determinants in higher dimensions. The whole analysis of this teaching and learning unit seems to offer valuable answers to the posed *ARQ3* (p. 63).

5.4 Analysis and Results Regarding ARQ4

In order the investigations to meet the answer of the auxiliary research question 4 (ARQ4) on how do students use and translate different modes of description and thinking of determinants in the designed DGE, students were given **four** tasks (see HLT in Section 4.4). The tasks are to be solved in a paper-pencil environment, after students have previously been exposed in a DGE. In this teaching and learning unit students' written works and their written mathematical language are in the focus of the analysis. The written mathematical language includes use of algebraic and geometric modes of description and translations from one into the other one, use of correct notations, mathematical explanations and argumentations of the solutions, verifications and generalizations.

5.4.1 Task 1. Parallelogram

In the **Task 1. Parallelogram** (in Section 4.4, p. 93 and worksheet in Appendix D) the algebraic representation of determinants is given and students are asked not only a) to calculate, but also to explain what is the geometric interpretation of the determinant $\begin{vmatrix} -4 & 4 \\ 3 & -1 \end{vmatrix}$. Moreover, students have to b) provide a geometric figure in the Cartesian coordinate system (given on the worksheet) to validate their answers to the previous question a). Thus students have to translate from arithmetic-algebraic into geometric mode of description of the given determinant. This means that in the requirement a) students make a *reflective connection*, because they are trying to explain the geometric mode of description with the use of algebraic language and in the requirement b) students make an *associative connection*, because they change one representation with another (according to [Hähkiöniemi, 2006], p. 40). Finally, they have to c) show that the area of the parallelogram equals absolute value of the determinant. All these requirements a), b) and c) are enclosed in a single task (see Task 1. Parallelogram in Section 4.3, p. 93 or worksheet in Appendix D).

In Task 1 a) a student calculates the value of the given determinant correctly (Figure 5.22) and notes that it is negative, equal to -8; and that the area of the parallelogram is 8 square units. The student is able to state a general conclusion by writing "the determinant is a number, which is the area of the figure in the plane" (Figure 5.22), but forgetting to mention the sign of the determinant[7]. In Task b) the student provides a good geometric mode of description, the parallelogram spanned by the vectors $\begin{pmatrix} -4 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, (Figure 5.22). The visualization shows student's proof in the requirement c), by use of geometric intuition in forming two rectangles consisted of two right-angled triangles, whose areas are denoted by A_1 and A_3 by the student (well drawn by red and green colors on the Figure 5.22). The student also uses two other rectangles with the same area A_2 (drawn with blue color on the Figure 5.22) for the proof. The general idea $A_{GES} - (|A_1| + |A_2| + |A_3|) = 8 = A$ (Figure 5.23[8]) (the student uses the symbol for absolute value for areas) gives a correct argumentation. This strategy, with the use of rectangles (denoted by the student as A_{GES} and A_2 on Figure 5.22 and Figure 5.23) and right-angle triangles (denoted as A_1 and A_3 on Figure 5.22 and Figure 5.23) is the most common geometric mode of description in the students' solutions of this task.

In the solution a), the second student makes a clear distinction that a given determinant may have a negative value, while the area of a plane geometric figure is always positive "because there is no negative area" (Figure 5.25), but student's argumentation why the determinant has a negative value is wrong, which can been seen by the student's claim: "The minus just appears because one vector is on the left side of the y-axis" (Figure 5.25). On one hand side this argumentation may be the reason why the student decided to use translation and work on solving the problem when the figure is completely located in the first quadrant. On the other hand side, this student's geometric mode of description goes beyond the expectations,

[7]This coincides with the identified research problem and located student difficulty (ii') in Subsection 3.1.3, page 61.

[8]A_{GES} means total area in English.

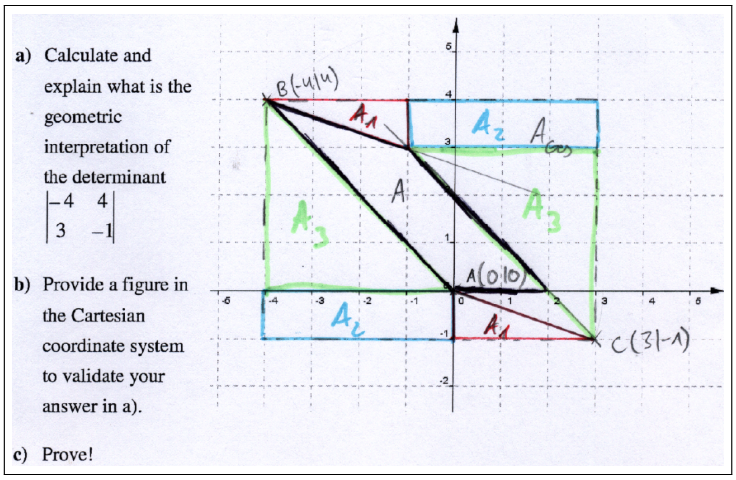
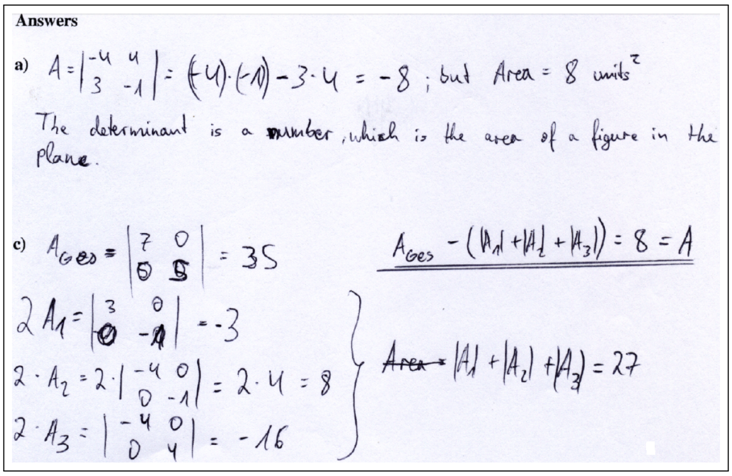


Figure 5.22: First Student's Geometric Representation in Task 1. Parallelogram


Answers

a) $A = \begin{vmatrix} -4 & 4 \\ 3 & -1 \end{vmatrix} = (-4)\cdot(-1) - 3\cdot 4 = -8$; but Area = 8 units^2

The determinant is a number, which is the area of a figure in the plane.

c) $A_{Ges} = \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix} = 35$ $\qquad$ $A_{Ges} - (|A_1| + |A_2| + |A_3|) = 8 = A$

$2A_1 = \begin{vmatrix} 3 & 0 \\ 0 & -1 \end{vmatrix} = -3$

$2 \cdot A_2 = 2 \cdot \begin{vmatrix} -4 & 0 \\ 0 & -1 \end{vmatrix} = 2 \cdot 4 = 8$ $\qquad$ ~~Area~~ $|A_1| + |A_2| + |A_3| = 27$

$2 \cdot A_3 = \begin{vmatrix} -4 & 0 \\ 0 & 4 \end{vmatrix} = -16$

Figure 5.23: First Student's Algebraic Representation and Proof in Task 1. Parallelogram

because (s)he evokes her/his rich concept images on vectors and applies this knowledge into solving the problem by using translation. Once (s)he has well drawn the parallelogram spanned by the vector rows in the given determinant, (s)he translates the parallelogram into the first quadrant where (s)he completes the figure with additional geometric figures (rectangles, with yellow color and right-angled triangles, with green and orange color, Figure 5.24). Further on, the student is capable of determining the new components of the vectors and applying them as entries in the corresponding determinants for the areas (Figure 5.25, answer c)). It seems that this student's visualization is strongly influenced by the geometric mode of description of vectors and rich concept images of vectors as classes of arrows which are equal in length and have same direction and orientation.

The third student also provides a correct solution on the requirements a) and b), Figure 5.26, but compared to the previous two, it seems that this student has a stronger algebraic image and is more familiar with algebraic modes of description. It can be detected in student's use of general notation of determinants in the proof

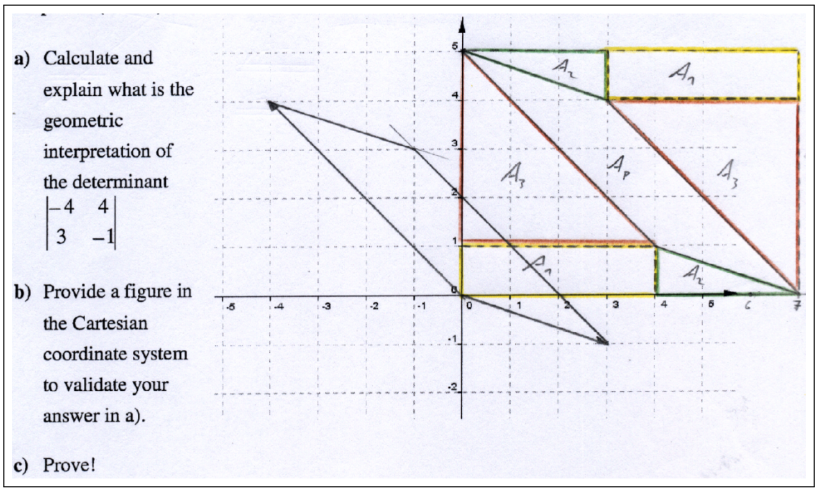
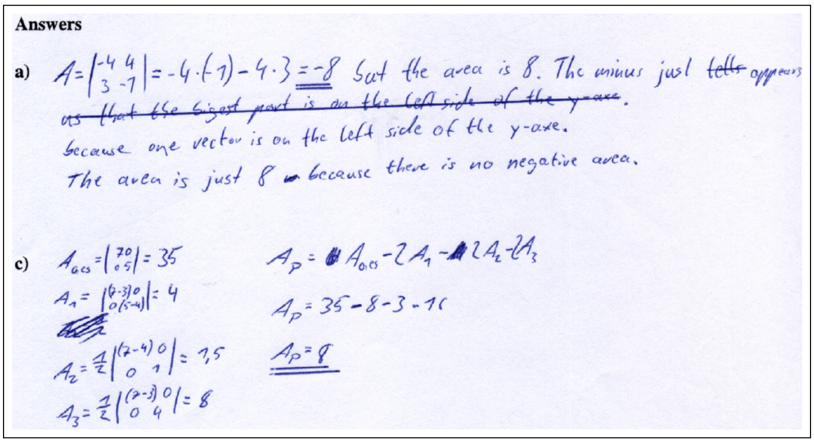


Figure 5.24: Second Student's Geometric Representation in Task 1. Parallelogram



Answers

a) $A = \begin{vmatrix} -4 & 4 \\ 3 & -1 \end{vmatrix} = -4\cdot(-1) - 4\cdot 3 = -8$ but the area is 8. The minus just ~~tells~~ appears ~~as that the biggest part is on the left side of the y-axe~~.
because one vector is on the left side of the y-axe.
The area is just 8 because there is no negative area.

c) $A_{0,0} = \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix} = 35$ $A_p = A_{0,0} - 2A_1 - 2A_2 - 2A_3$

$A_1 = \begin{vmatrix} (7-3) & 0 \\ 0 & (5-4) \end{vmatrix} = 4$ $A_p = 35 - 8 - 3 - 16$

$A_2 = \frac{1}{2}\begin{vmatrix} (7-4) & 0 \\ 0 & 1 \end{vmatrix} = 1,5$ $A_p = 8$

$A_3 = \frac{1}{2}\begin{vmatrix} (7-3) & 0 \\ 0 & 4 \end{vmatrix} = 8$

Figure 5.25: Second Student's Algebraic Representation and Proof in Task 1. Parallelogram

of task c) (Figure 5.27). The student provides a correct algebraic mode of description with general notation and then specifies it for the given particular problem by substituting the corresponding number values and obtained a correct result.

What differs from the previous two argumentations is the direct use of knowledge about areas of rectangles. Namely the student recalls the formula for calculation of an area of a rectangle (equal to the product of the lengths of the sides) and uses it correctly instead of calculating areas of rectangles with the aid of determinants (detected in three other students' solutions). The student denotes all corresponding calculations in the general notation, in the specific notation and in the geometric mode of description with matching colors (green, pink and black), thus the student establishes links between both modes of description. This shows student's capability to connect her/his knowledge in elementary geometry and the new knowledge about determinants.

Students' use of matching colors (Figure 5.22, Figure 5.24, Figure 5.26 and Figure 5.27) may have been stimulated by the use of matching colors of corresponding

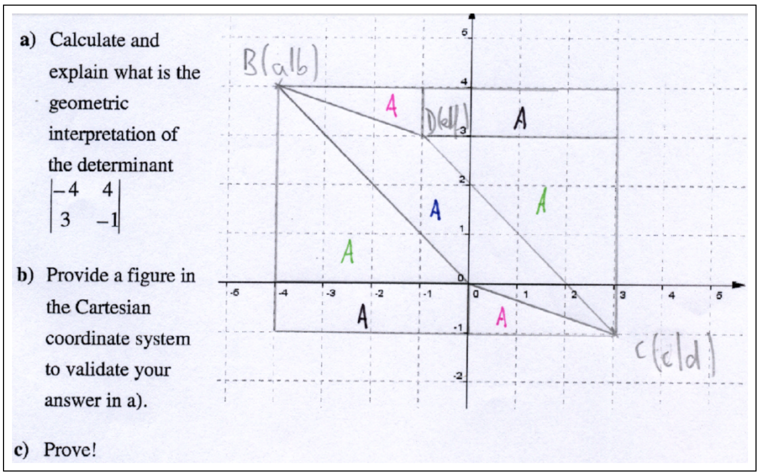


Figure 5.26: Third Student's Geometric Representation in Task 1. Parallelogram

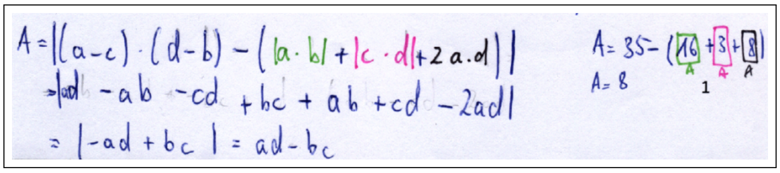


Figure 5.27: Third Student's Algebraic Representation and Proof in Task 1. Parallelogram

entities in different modes of description on the designed Applets 3 and 4.[9]

Unlike to the previous three students' solutions, whose geometric modes of description are consisted of a parallelogram spanned by the vector rows in the given determinant $\begin{pmatrix} -4 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, another student's visual representation utilizes vector columns of the given determinants, so $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$, for spanning the parallelogram (Figure 5.28). This student has also written both possibilities for choosing the coordinates of the vertices B and C of the parallelogram (below the drawing on the same Figure 5.28), which confirms that this student understands the property that a determinant of a transposed matrix is equal to the determinant of a matrix itself. Further on, the student correctly applies the property in a problem solving situation.

In contrast to the previous four students' solutions, the next one shows inadequate geometric mode of description (Figure 5.29). Namely, the student chooses the diagonal entries of the given determinant instead of the column or row entries, so uses the

[9]Matching colors for linking and highlighting important relationships between different modes of description is a design measure which supports learning ([Bodemer et al., 2004]). This colourful effect, as a so called split-attention effect ([Mayer & Moreno, 1998]) of the designed Applets focuses students' cognitive load on the desired part of the content and prevents overburdens of the learning capacities.

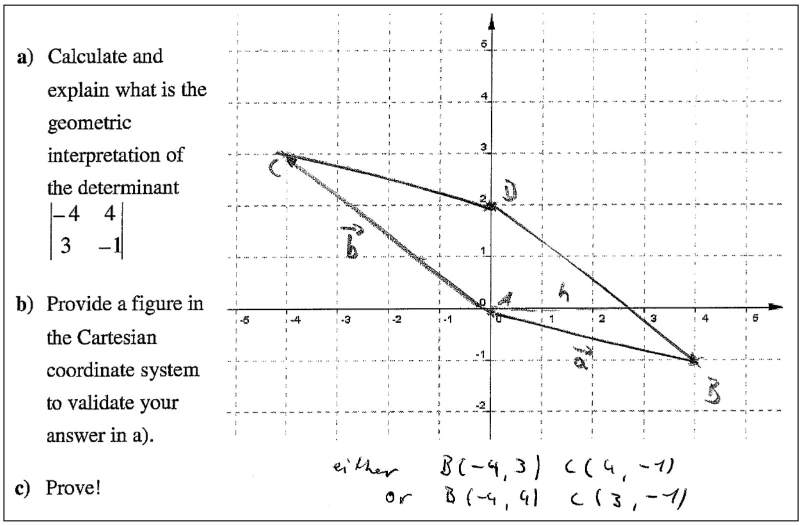


Figure 5.28: Fourth Student's Geometric Representation in Task 1. Parallelogram

vectors $\begin{pmatrix} -4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ to form the parallelogram[10]. However, the calculation of the value of the given determinant is correct (Figure 5.30). In the requirement c) this student evokes her/his concept images on parallelogram and recalls the formula for calculating its area as a product of the length of a side and the corresponding altitude (this strategy was used by one more student). The student correctly calculates length of the side, by using the Pythagorean Theorem for a well chosen right-angle triangle (in the student's own drawing, Figure 5.30), but could not determine the corresponding altitude. It seems that besides some difficulties with establishing connections between the arithmetic-algebraic and the geometric modes of description and between the concept of determinant and the elementary geometry concepts, this student has a good potential to overcome them by further tasks.

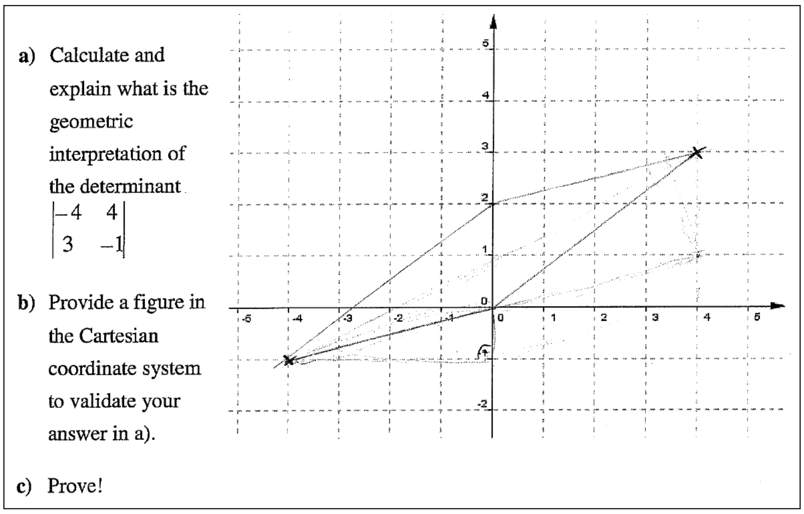


Figure 5.29: Fifth Student's Geometric Representation in Task 1. Parallelogram

[10]Choosing appropriate entries was part of the discussion in lines from [34] to [40], p. 121

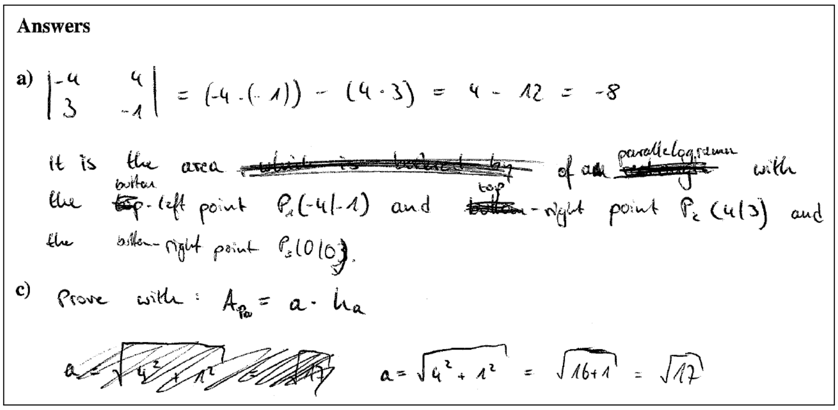


Figure 5.30: Fifth Student's Algebraic Representation and Proof in Task 1. Parallelogram

An interesting case for analysis is the solution given by the sixth student who correctly calculated the value of the determinant and provided a correct geometric mode of description, but was not capable to provide a symbolic notation to state the proof in the requirement c) (Figure 5.31 and Figure 5.32). Instead (s)he offers an explanation which confirms that (s)he has learned the fact that two vectors span a parallelogram with an oriented area, as (s)he says: "an area with an "oriented" value". The problem is student's argumentation for the negative value of the determinant, because "a bigger part is located in the "negative" area of the coordinate system" [meaning the second quadrant]. This is similar misconception as located in the second student's solution.

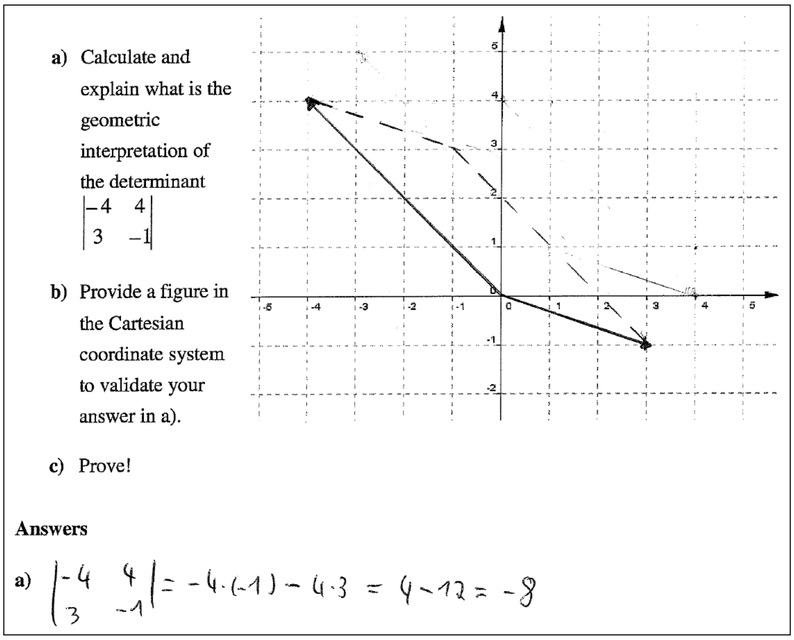


Figure 5.31: Sixth Student's Geometric Representation in Task 1. Parallelogram

Completely opposite from all previous successful students' attempts for geometric interpretation of determinants, one incorrect student's geometric mode of description which was detected is the following. The student makes a mistake in the calculation

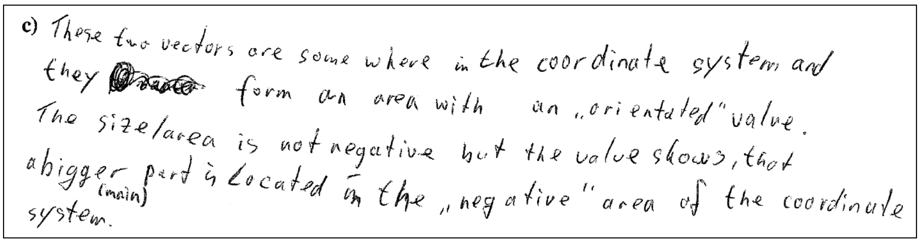

c) These two vectors are some where in the coordinate system and they form an area with an „orientated" value. The size/area is not negative but the value shows, that a bigger (main) part is located in the „negative" area of the coordinate system.

Figure 5.32: Sixth Student's Algebraic Representation and Proof in Task 1. Parallelogram

of the value of the given determinant and moreover (s)he is not able to determine the appropriate vectors and span the parallelogram (demonstrates uncertainty that the figure is a parallelogram, Figure 5.21) and does not offer a proof of her/his statement that "the area is 8" (Figure 5.22). Thus, the student makes an incorrect choice of the vertices of the parallelogram, namely chooses diagonal entries of the given determinant instead of the row (column) entries. This students' poor performance could be attributed to her/his lack of previous knowledge in elementary and Analytic geometry, more preciously coordinates of points and components of vectors (later proved in the next task) and because of that cannot be considered as a misconception of determinants.

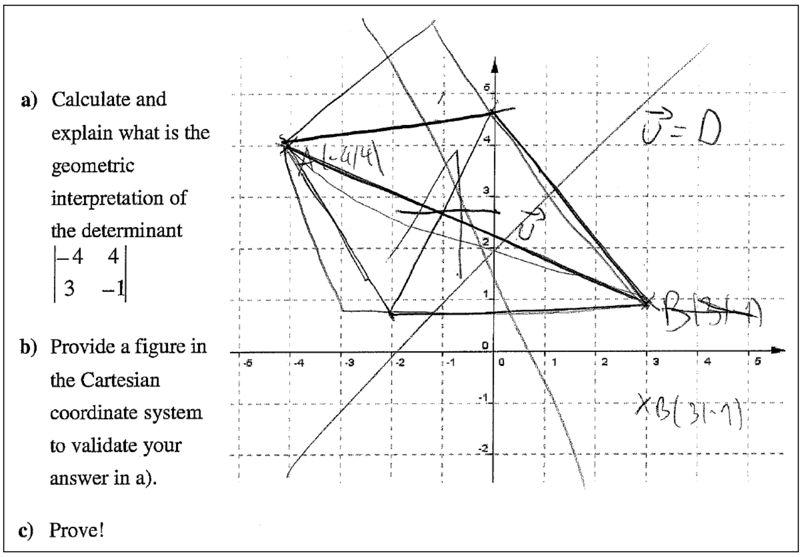


Figure 5.33: Seventh Student's Geometric Representation in Task 1. Parallelogram

The general conclusions arising from this part of the learning trajectory are that most of the students are able not only to calculate values of determinants, thus to deal with arithmetic-algebraic mode of description (with an exception of the last student), but have also successfully understood geometric interpretation of determinants, thus that absolute value of the given determinant equals the area of a corresponding parallelogram (again all students with the exception of the last one). These achievements may be due to the use of the visualizations through the dynamic Applets for the Axioms 3a and 3b and students' active participation. However, one

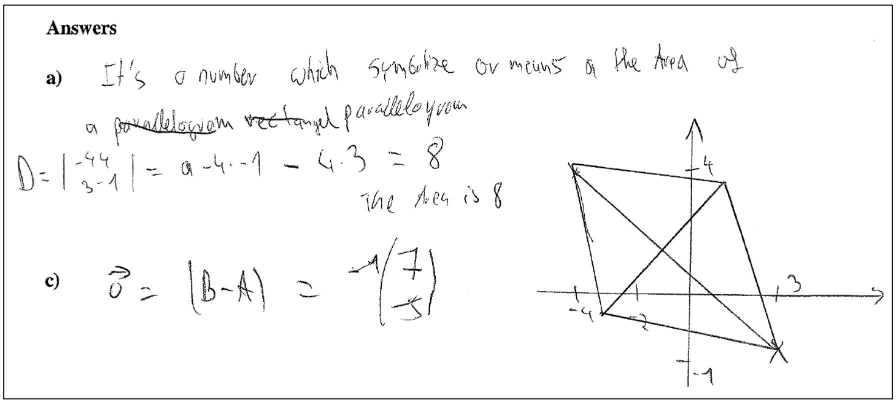


Figure 5.34: Seventh Student's Algebraic Representation and Proof in Task 1. Parallelogram

misconception occurs in the analysis of students' works. A misconception which occurs in two student's answers (the second and the sixth) is related to the plus and minus signs of determinants, which neither depends on the quadrant of the Cartesian coordinate system where the parallelogram is located, nor on the quadrants in which each of the vectors spanning the parallelogram is located, as students have thought. This misconception occurred when students were working in a paper-pencil environment (without any use of the applets for this particular problem). These particular students were advised to further work with the applets in order to avoid the same obstacle about orientation, as was identified with the university students (identification of the research problem, students' difficulty (ii') in Subsection 3.1.3, p. 61). For the proof in c), students used two strategies. Most of them evoked formulas for calculating areas of rectangles and right-angle triangles and applied them as one strategy for proving and the rest of the students evoked the formula for calculating areas of parallelograms as a product of the length of a side and corresponding altitude. The first strategy showed to be more successful. A strategy using exclusively right-angle triangles exists and is presented on Figure 5.35, but none of the students came up with such an idea.
A cumulative overview on students' answers to this Task is the following. Students' solutions on both requirements in a): one, *calculate* and two, *explain* characterize an interaction between two kinds of symbolic representations: *formal* (definition) trough mathematics language and *verbal* (description) through natural language, as suggested by [Tall, 1995]. These two symbolic representations increase students' conceptual understanding, because they not only carry out sequenced procedures as routines (calculate following a single pattern), typical for procedural knowledge, but also mental processes which involve symbols for a new concept as determinants.

5.4.2 Task 2. Triangle

Further investigations on students' algebraic and geometric modes of description and language, as well as connections between more concepts continue in the **second**

a) Calculate and explain what is the geometric interpretation of the determinant $\begin{vmatrix} -4 & 4 \\ 3 & -1 \end{vmatrix}$

b) Provide a figure in the Cartesian coordinate system to validate your answer in a).

c) Prove!

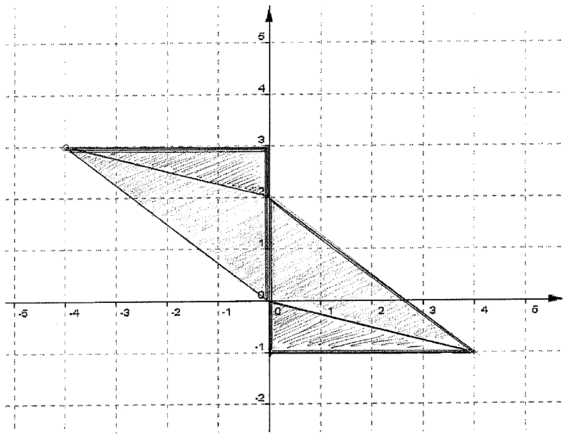


Figure 5.35: Possible visualization for Task 1

Task called **Triangle** (p. 94 or Appendix D) which has also three requirements. The aim of the task is to show students that determinants can also be used for calculation of oriented areas of other geometric figures, besides parallelograms, namely triangles. The new problem situation, compared to the previous task, is constructed such that none of the vertices of the triangle is located at the origin of the coordinate system and a matrix is not given, but should be formed by the students. So, they have to provide both the geometric and the arithmetic-algebraic modes of description for determinants when only coordinates of points are given. How students react in this situation and what kind of solving strategies they use can be seen from the following elaboration.

The first student (Figure 5.36) uses translation (was not the case in the previous task) as a solving strategy for this problem. The student translates the triangle (after correctly sketching the initial triangle in the coordinate system) for a vector $\begin{pmatrix} -5 \\ 0 \end{pmatrix}$, which means places the vertex $A(5,0)$ at the origin of the coordinate system (Figure 5.36). After that, the student correctly determines the new coordinates of the vertices B and C, in other words components of the vectors $\begin{pmatrix} -6 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -8 \\ -2 \end{pmatrix}$ spanning the parallelogram whose half area equals the area of the given triangle. Same as in the previous task (Figure 5.22 and Figure 5.23) the student uses vector rows in the determinant. In the requirement c) of this task, the student generalizes the area for a triangle when none of its vertices is at the origin, by correct entries in the determinant (only forgetting the coefficient $\frac{1}{2}$, although it appears in the solution a)).

The second student sketched the triangle with the given coordinates and then sketched a congruent triangle $A'B'C'$ (Figure 5.37). Compared to the previous student's solution, here it is not stated that translation is used, nor a vector is mentioned. It seems like congruence of triangles is the chosen solving-strategy and

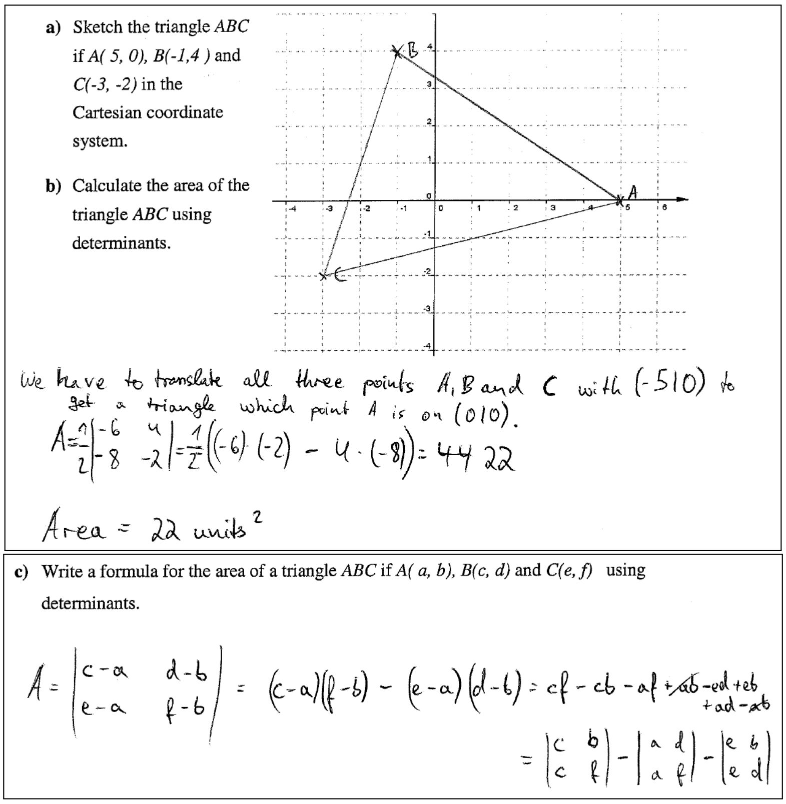


Figure 5.36: First Student's Solution of Task 2. Triangle

not translation. The algebraic solution shows application of calculations of areas of a rectangle and three right-angle triangles, each one obtained by determinants (although there exist simpler paths). These calculations later reflect in the general solution offered by the student.

The third student, like the first one, uses translation as a solving strategy for this task, just the vector of translation is different, $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (Figure 5.38). So, this student translates the triangle such that the vertex C coincides the origin in the coordinate system. The decision on the vector of translation, whether the vertex A, B or C will coincide the origin, of course, does not affect the final solution. This student's solution shows her/his very good application of vector addition, both in algebraic and geometric language (all vertices of the triangle are translated by the same vector $\vec{v}$. For the generalization in the requirement c) the student also uses a correct arithmetic-algebraic language. The only mistake in the solution is the lack of the one half for the area of the triangle.

The first two requirements in the task a) and b) are completely correctly solved by the fourth student (Figure 5.39) and a successful generalization is established in the requirement c).

The fifth student provided similar solution (Figure 5.40) as the fourth student, short, clear and correct in both geometric and arithmetic-algebraic mode of description.

The last student, who gave wrong geometric mode of description on the previous

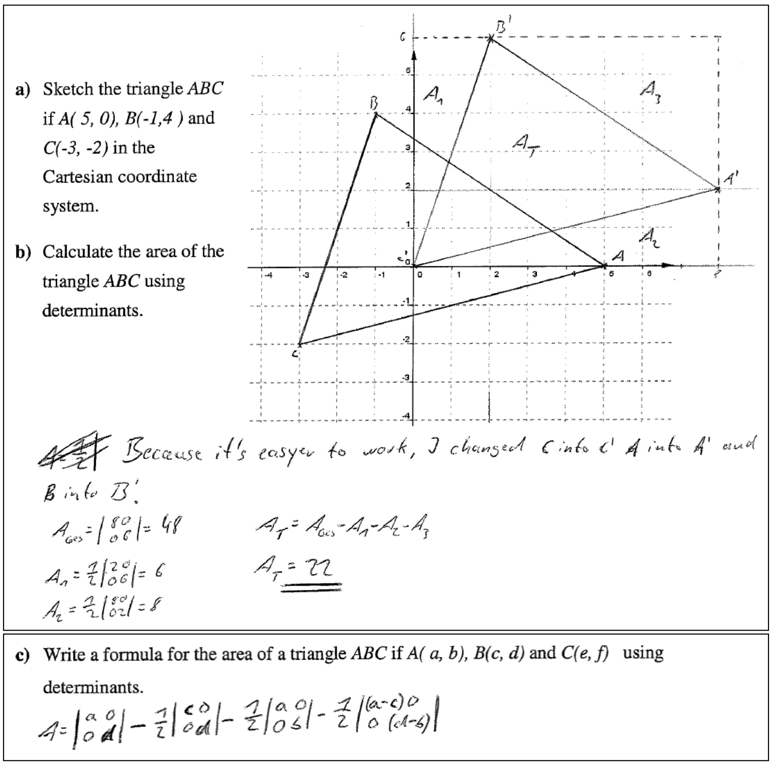


Figure 5.37: Second Student's Solution of Task 2. Triangle

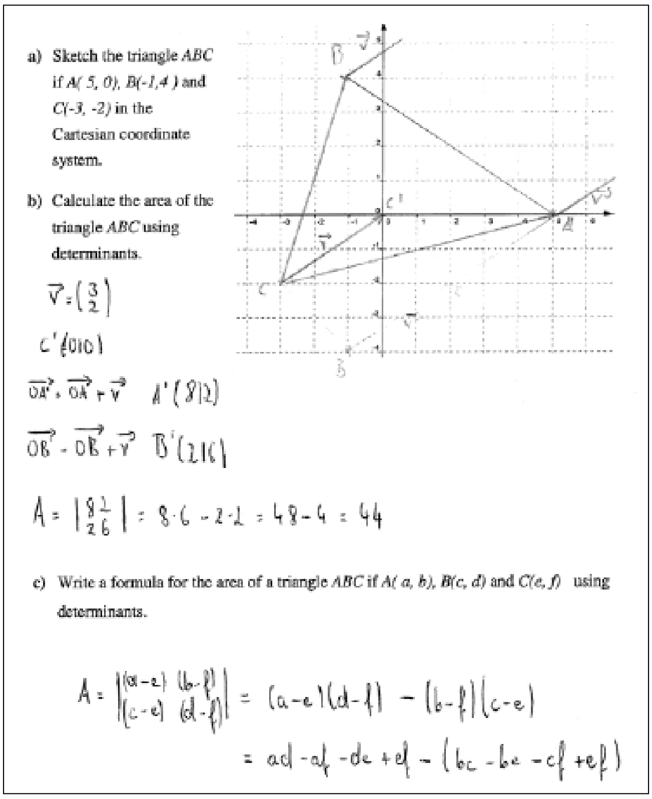


Figure 5.38: Third Student's Solution of Task 2. Triangle

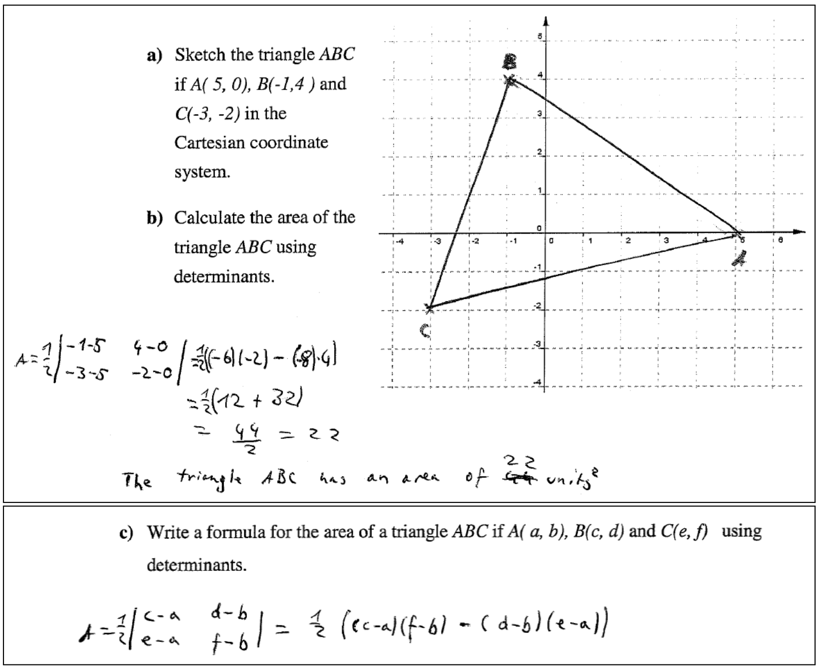


Figure 5.39: Fourth Student's Solution of Task 2. Triangle

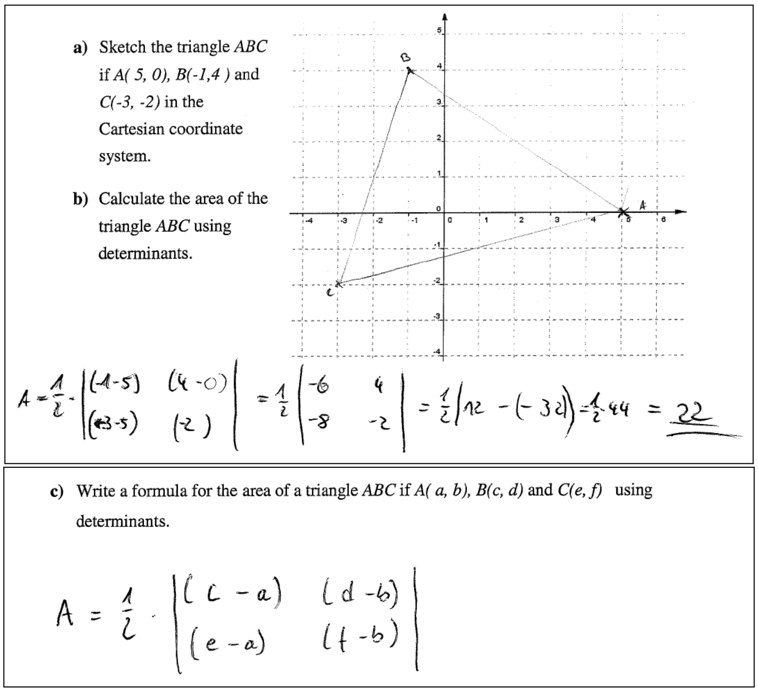


Figure 5.40: Fifth Student's Solution of Task 2. Triangle

Task 1 (Figure 5.33 and Figure 5.34), also demonstrates problems in the visualization of this task. There is a wrong placement of coordinates of points on both x- and y-axes (Figure 5.41). This may be the argumentation why this student's mistakes in geometric modes of description in both tasks cannot be considered as misconceptions of determinants, but rather as a lack of knowledge in vectors and elementary Analytic geometry.

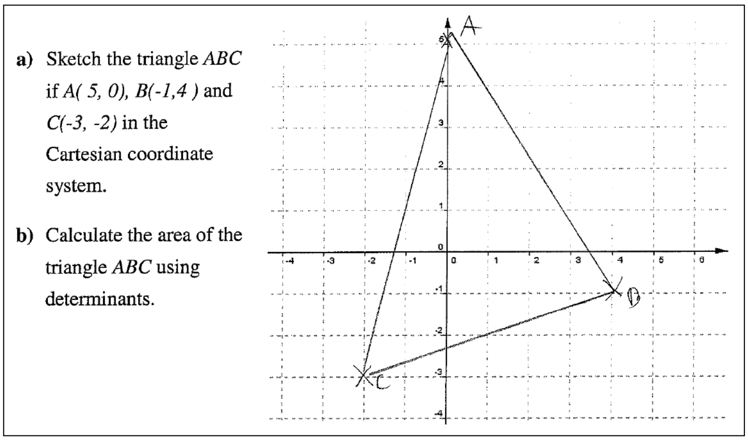


Figure 5.41: Sixth Student's Solution of Task 2. Triangle

Other suitable students' solution (Figure 5.42) uses another strategy, without translation or congruence of triangles, but with direct step by step solution by implementation of knowledge from elementary geometry with rectangles and right-angle triangles (which is also the most common student's strategy). This student does not recall general statement for oriented areas and determinants as a conclusion from the previous task with the parallelogram which can be implemented in this task. Although not so elegant, this solution is also correct.

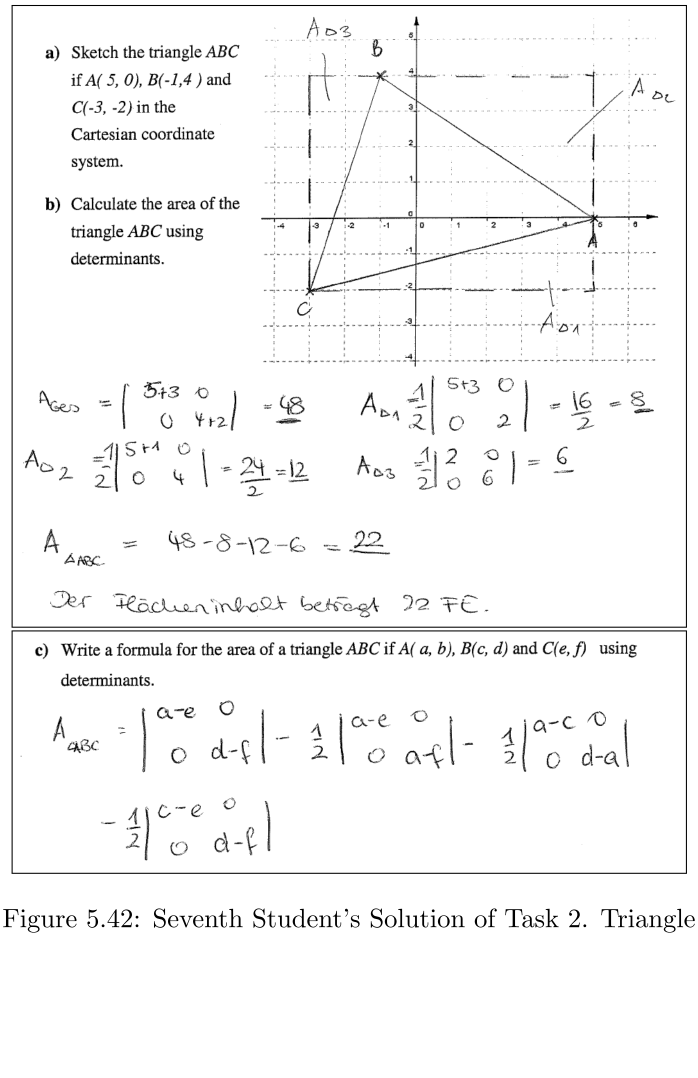


Figure 5.42: Seventh Student's Solution of Task 2. Triangle

Shortly summarized, in this teaching unit students implemented associative connections ([Hähkiöniemi, 2006]) between two modes of description and language: geometric and arithmetic-algebraic. From the previous analysis, it seems that students are capable to generalize their arithmetic-algebraic modes of description with the aid of three solving strategies: translations, congruence of triangles, and rectangles and right-angle triangles.

5.4.3 Task 3. Trapezoid

In the **third Task. Trapezoid** (p. 94 or Appendix D), a trapezoid is given in the Cartesian coordinate system, thus the geometric representation is given and students are asked to provide an arithmetic-algebraic mode of description which corresponds to the area of the given trapezoid. In comparison with the previous two tasks, which mainly asked for a translation from algebraic to geometric mode of description, in this task students have to translate vice versa, from geometric to algebraic mode of description. They are encouraged to use their own strategy in solving the problem. A successful solution of a student evoking the result from the first task is presented on Figure 5.43. This particular student practically separates the given trapezoid into two triangles and for calculating the area of each of these triangles the student applies half of the determinant corresponding to the appropriate parallelogram.

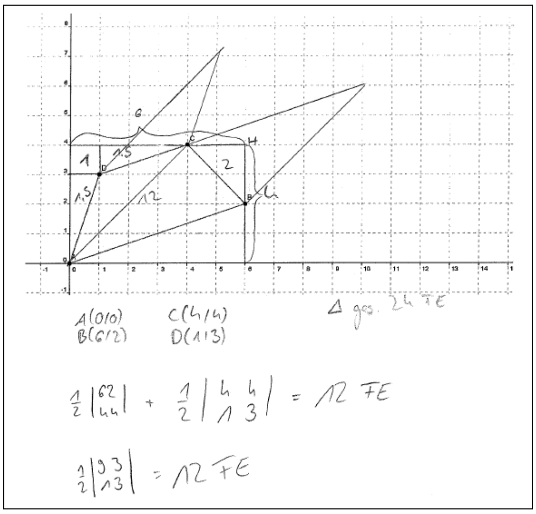


Figure 5.43: Student's Solution of Task 3. Trapezoid using Triangles

Although elementary geometry provides efficient formulas for calculating areas of trapezoids, students' performance on this task shows that (s)/he successfully adds two determinants and does not mix this operation with addition of matrices, for example.

One student's solution which deserves attention is the one presented on Figure 5.45. It may be classified as going beyond because this student went a step further than the requirements in the task by trying to offer a general solution which will include

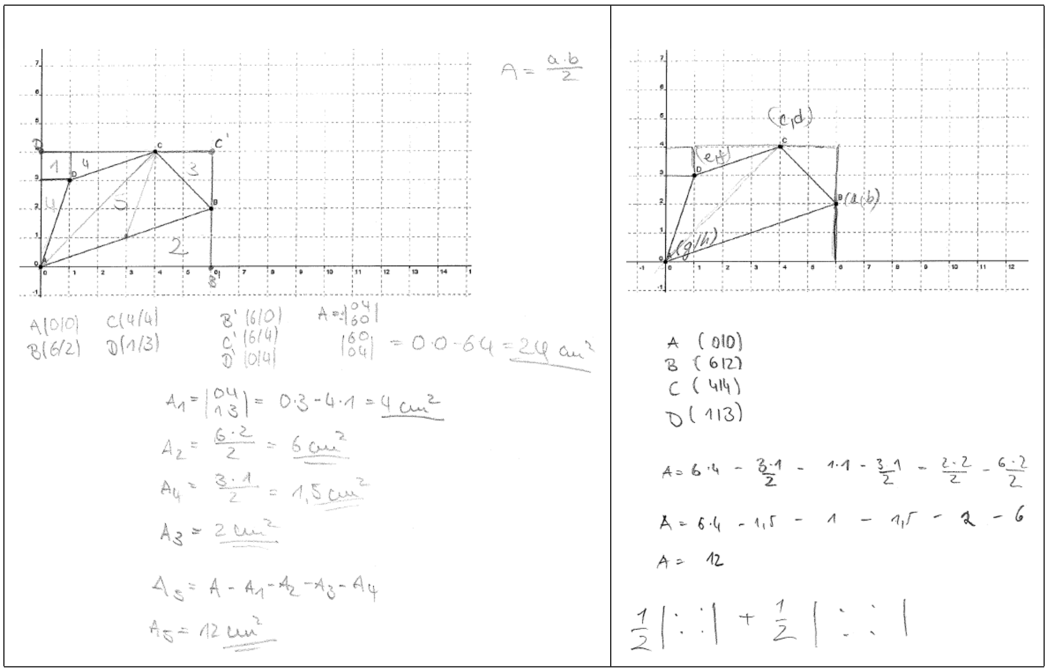


Figure 5.44: Students' Solutions of Task 3. Trapezoid using Rectangles

the given trapezoid as a special case. For this attempt, the student used an extra paper and after several trials successfully sketched a trapezoid with vertices $A(a,b)$, $B(c,d)$, $C(e,f)$ (Figure 5.45). Afterwards, (s)he additionally draw a congruent trapezoid such that they both form a parallelogram spanned by vectors $\overrightarrow{ED} = \begin{pmatrix} g \\ h \end{pmatrix}$ and $\overrightarrow{EA} = \begin{pmatrix} a \\ b \end{pmatrix}$ (according to the student's sketch). Thus, the required area of the given trapezoid, denoted as A_2 equals half of the area of this parallelogram, denoted as A_1 by the student. This forms a clear picture that student's visualization and geometric mode of description is quite prosperous. It may also be speculated that this idea may have been gained from elementary geometry courses about areas of quadrilaterals in grades 7 or 8. A problem, however, comes when (s)he has to translate into the arithmetic mode of description when determining the final determinant whose absolute value equals the area of the obtained parallelogram. Instead of $A = \frac{1}{2}\begin{vmatrix} a & b \\ c+(e-a) & d+(f-b) \end{vmatrix}$, the student wrote $A = \frac{1}{2}\begin{vmatrix} a & b \\ c\frac{e}{a} & d\frac{f}{b} \end{vmatrix}$ (on the bottom of Figure 5.45), thus switched the operations addition with multiplication and subtraction with division.

In the concluding paragraph about students' outcomes of this teaching and learning unit, it may be said the following. The technology to paper-pencil transfer of knowledge is beneficial in the following sense.

> Learning by actively producing an integrated format tends to be more successful than learning by merely reconstructing an already integrated format ([Bodemer et al., 2004], p. 330).

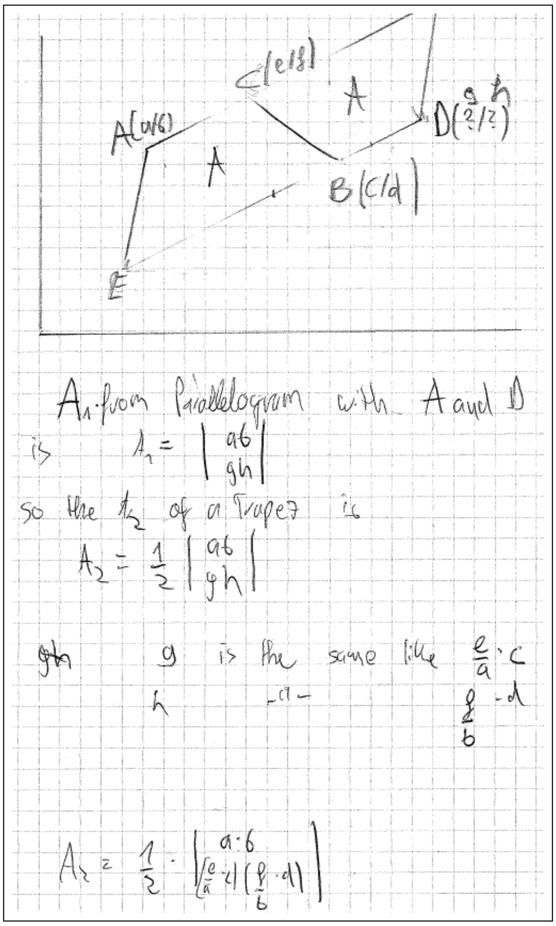


Figure 5.45: Student's Solution of Task 3. Trapezoid using a Parallelogram

Thus, students have first been "reconstructing an already integrated format" according VDS and discussions on the guiding questions in the DGE and then "actively producing" in the paper-pencil counterpart environment. This tidiness between the two counterparts of the suggested DGE appears evident. Namely, most of the students have been strongly stimulated by the dynamic properties (simultaneous presence and variation of both modes of description) and design measures (for example matching colors). Due to the exposure to the designed learning materials in the previous teaching unit, the students successfully reflected upon and created outcomes as colourful static geometric visualizations (Figures 5.22, 5.24, 5.26, 5.37a), 5.38a), 5.39a), 5.40a), 5.43, 5.44 and 5.45) and accompanying elegant arithmetic-algebraic solutions (Figures 5.23, 5.25, 5.27, 5.36b), c), 5.37b), c), 5.38b), c), 5.39b), c), 5.40b), c), 5.43, 5.44 and 5.45). While students learned what is and what is not a determinant (guiding feature 1, p. 37) in the previous teaching unit, (an oriented area instead of just an area through a discussion about the importance of the sign in lines [50] to [56]), and also faced multiple modes of determinants' descriptions (guiding feature 3, p. 37), they now successfully manipulated these modes in problem solving situations (guiding feature 5, p. 37). Thus, conceptual understanding has been growing on local level, i.e. horizontally. Meanwhile, students applied their previous knowledge from elementary geometry (areas of triangles, parallelograms, trapezoids), thus lower secondary school, in connection with the new topic about determinants in upper secondary school. This relates to the global vertical individual's development of conceptual understanding, which together with the above discussion seems to offer answers to the ARQ4, p. 63.

5.4.4 Task 4. Locus

In order to solve **Task 4. Locus** (p. 94 or Appendix D) students work with the Applet 4 (p. 95) and are encouraged to discuss the solution in pairs. The initial position of the applet is set such that the given triangle has an area of 16 squared units (same as the value of the displaced determinant), so students have to drag the point C in order to get an area of 24 squared units, while keeping the other two vertices of the triangle A and B fixed. Simultaneously they also have to focus on the entries of the determinant and its value. First, students discuss on the requirements a) and b) and suggest their discoveries for the coordinates of the point C lying on each of the axis. Correct answers $(0, 4)$ and $(-2, 0)$ are then written on the board by the instructor. Students continue to suggest other coordinates of the movable vertex C (not lying on the axes) they have discovered. Once they realize that there exist infinitely many possibilities and the locus is a line (answer of the requirement c), they try to find its equation. Students' reasoning while they work in pairs on this problem and hypothesize on the exact line can be tracked through the transcript excerpt of the recorded dialogue.

[1] S1: It must be parallel. [Meaning parallel to $\overline{AB}$] (Es muss parallel sein[11]).

[2] S2: No. I don't think so. (Nein. Ich glaube nicht).

[3] S1: I think it must be parallel. But, ... (Ich denke es muss parallel sein, aber die ...) [Changing the positions of C on the applet.]

[4] S2: Could you see? (Siehst du?) [Pointing on the screen, (Figure 5.46)].

[5] S1: That's right. Of course it must be parallel. [Placing his hand on the screen, Figure 5.47] That's right, because g and h must stay unchanged. And we also have another one $2x - 50$. (Stimmt, natürlich, es muss parallel sein. Stimmt, weil g und h sich nicht verändern und dass ... und dann haben wir andere $2x - 50$ auch).

[6] S2: No, no. $2x - 20$ (Nein, nein. $2x - 20$.)

[7] S1: Yes, $2x - 20$. (Ja, $2x - 20$.)

[11] Whenever students talked to each other while exploring the Applets, they used the German language.

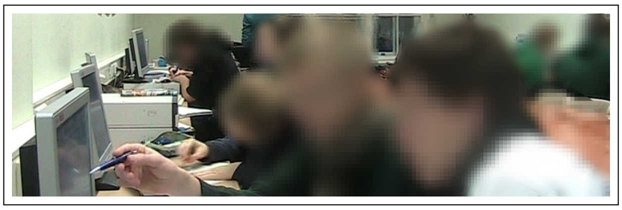


Figure 5.46: S2 Pointing on the Screen while Discussing Task 5 in the DGE

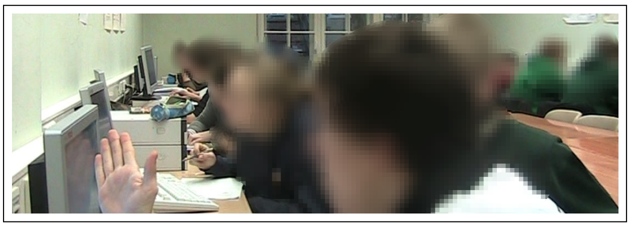


Figure 5.47: Placing his Hand on the Screen, S1 Explains That the Slope of Both Lines is the Same, thus 2.

These two students S1 and S2 not only have found the equation of the line passing through the point C, when C lies on the y-axis (or C lies on the x-axis) and parallel to the side AB of the triangle, but have also found the second solution. They were able to explain why there are exactly 2 lines parallel to AB which keep the area of the triangle ABC equal to 24 (because of preserving the altitude h of the triangle passing through the vertex C). Because of this heuristic discovery student S1 was elected (after students' work in pairs) to explain it in front of the classmates and present his geometric mode of description on the board, Figure 5.48 (besides on his worksheet).

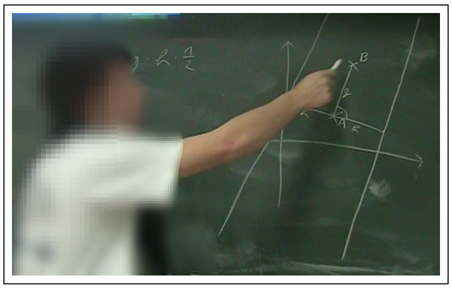


Figure 5.48: Student's 1 Geometric Mode of Description on the Board

Student's (S1) solution shows that he finds geometric language easier for expression. Namely, he easily visualized that all different positions of the vertex C providing the same area of the triangle ABC practically lie on two fixed parallel lines equally distant from the side AB of the triangle and sketched his own geometric visualization (Figure 5.48). However, he manifests certain difficulties with translating into analytic-algebraic mode of description or it appears incomplete on the worksheet (Figure 5.49).

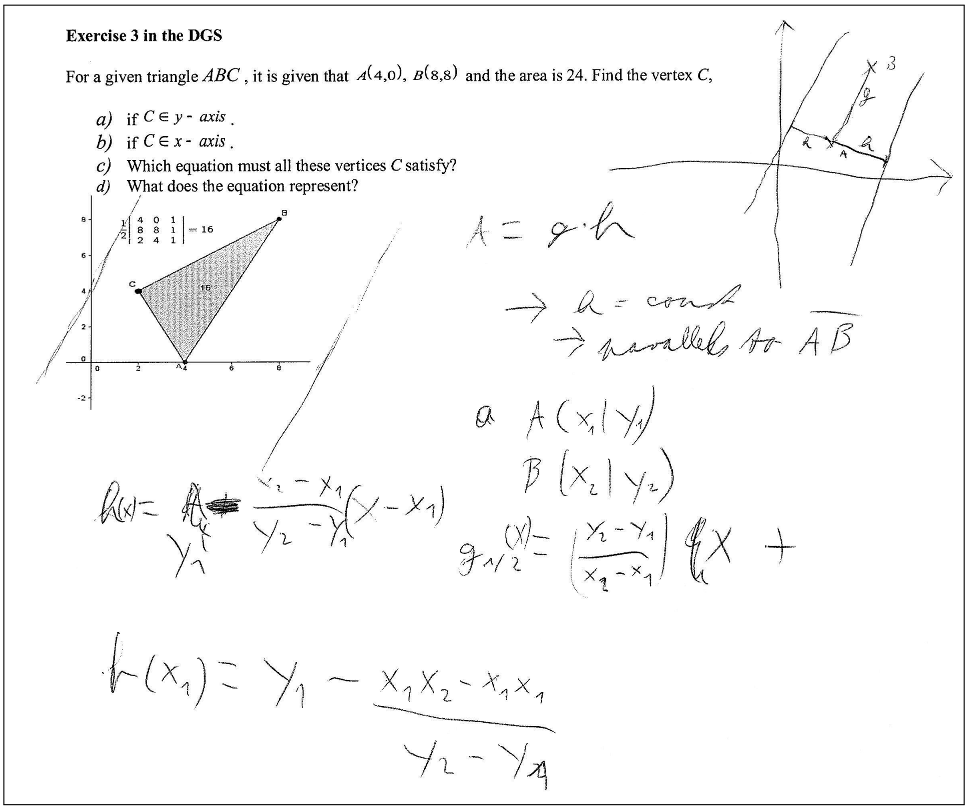


Figure 5.49: Student's 1 Analytic-algebraic Mode of Description

On the contrary of student's 1 geometric mode of description, student S3 provides appropriate arithmetic-algebraic mode of description of the same problem and is for this reason also elected by the instructor to show her solution. She uses the fact that the area of the triangle equals absolute value of the determinant being displaced on the applet. Thus, choosing each of the + and - signs of the determinant (values of 24 and -24, both providing the same area of the triangle ABC equal to 24), she proves the existence of exactly two lines and successfully transfers the algebraic description of determinants from the screen to the board, Figure 5.50 (and on paper, Figure 5.51) obtaining equations of both lines.

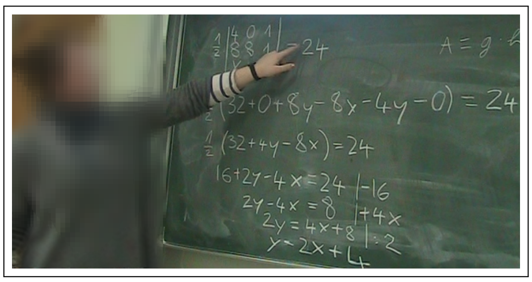
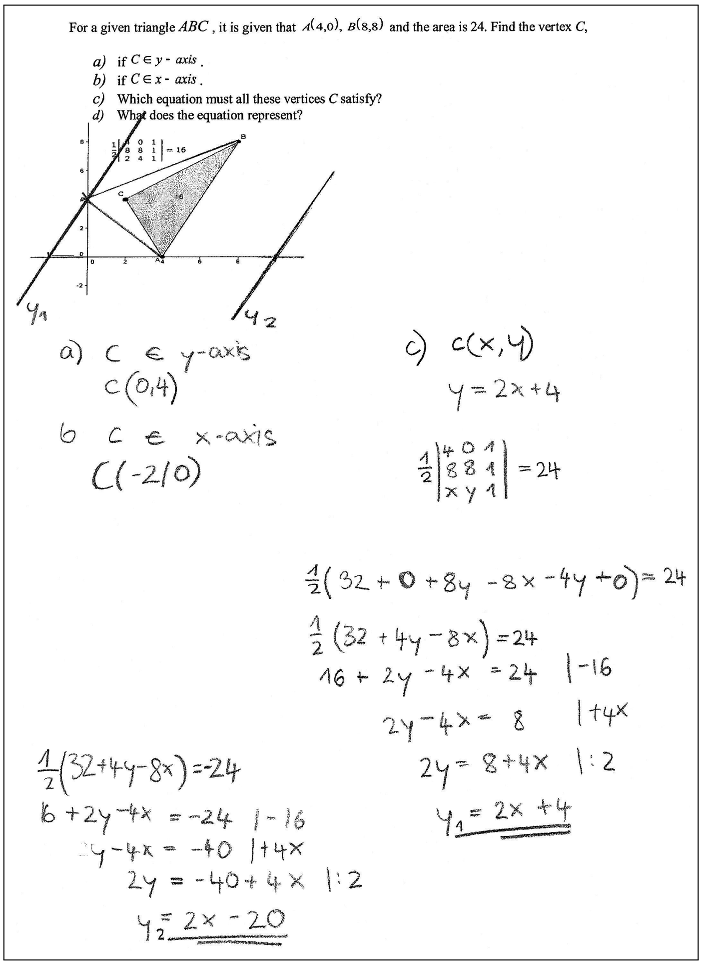


Figure 5.50: Student's 3 Arithmetic-algebraic Mode of Description in Task 5 on the Board

Student's 3 solution shows how she connected the concept of a determinant, when its value (a real number) changes from presenting an area of a triangle into an equation


For a given triangle ABC, it is given that $A(4,0)$, $B(8,8)$ and the area is 24. Find the vertex C,

a) if $C \in y$ - *axis*.
b) if $C \in x$ - *axis*.
c) Which equation must all these vertices C satisfy?
d) What does the equation represent?

$\frac{1}{2}\begin{vmatrix} 4 & 0 & 1 \\ 8 & 8 & 1 \\ 2 & 4 & 1 \end{vmatrix} = 16$

y_1 y_2

a) $C \in y$-axis
$C(0,4)$

b $C \in x$-axis
$C(-2/0)$

c) $C(x,y)$
$y = 2x + 4$

$\frac{1}{2}\begin{vmatrix} 4 & 0 & 1 \\ 8 & 8 & 1 \\ x & y & 1 \end{vmatrix} = 24$

$\frac{1}{2}(32 + 0 + 8y - 8x - 4y + 0) = 24$
$\frac{1}{2}(32 + 4y - 8x) = 24$
$16 + 2y - 4x = 24 \quad |-16$
$2y - 4x = 8 \quad |+4x$
$2y = 8 + 4x \quad |:2$
$\underline{y_1 = 2x + 4}$

$\frac{1}{2}(32 + 4y - 8x) = -24$
$16 + 2y - 4x = -24 \quad |-16$
$2y - 4x = -40 \quad |+4x$
$2y = -40 + 4x \quad |:2$
$\underline{y_2 = 2x - 20}$

Figure 5.51: Student's 3 Geometric and Arithmetic-algebraic Mode of Description in Task 5 on Paper

of a line, by choosing the coordinates of the vertex C from concrete values (the starting position $C(2,4)$) into any real numbers $C(x,y)$[12]. This is a confirmation that she managed very well to change her concept image of a 3 by 3 determinant representing a real number into a concept image of a 3 by 3 determinant representing a line. Her rich concept images include the geometric aspect, as shown on the Figure 5.51. From her geometric mode of description it also seems that she understands that both points in a) and b) lie on the same line y_1. Thus, the conclusion would be that the geometric mode of description and interpretation of determinants contribute to connecting different concepts, in this particular example: determinants, points, lines and triangles.

Third approach on the same problem was offered from student S4, who wrote the following (Figure 5.52):

[12]This conceptual change is analogue as, for example, when the mathematical statement $1+2=3$ changes into a linear equation in one variable $x+2=3$, by substituting a number with a variable; or furthermore into a linear equation in two variables $x+y=3$.

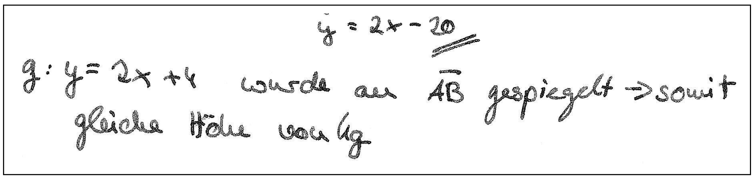


Figure 5.52: Student's 4 Answer to the Task 5.c)

The English translation of the student's answer is: the line $g : y = 2x + 4$" could be obtained by reflection (symmetry) with respect to $\overrightarrow{AB}$ thus, at the same altitude h_g"; or by a "parallel shift" (Task d), Figure 5.53), i.e. translation.

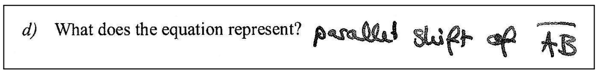


Figure 5.53: Student's 4 Answer to the Task 5.d)

This student used the algebraic notation, similar as student's 3, but connected the geometric interpretation with completely other concepts, namely symmetries. Even though (s)he has made some mistakes in correct written math language (for example, she wrote symmetry with respect to a segment $\overline{AB}$ instead of a line AB and for example, "parallel shift of AB", probably meaning translation by a vector, but the vector of translation is not specified), it is of great interest how (s)he connected the solution with a concept as symmetry. Similar to this reasoning is also student S5 (Figure 5.54).

Both of these two students S4 and S5 have developed geometric mode of thinking as their explanations testify, but neither of them provided a geometric mode of description on the worksheet.

Student S6 derives her conclusions based on the special cases in the requirements a) and b) in order to find the equation required in c), Figure 5.55. The student states that (s)he used both points on each of the axis to determine the slope and the y-intercept, denoted as m and n in student's solution, respectively (same Figure 5.55). Further on in d), (s)he calculates the value of the determinant and obtains the same equation $y = 2x + 4$ as in c) and the second equation $y = 2x - 20$. This shows that besides appropriate arithmetic-algebraic mode of description (s)he can also think in an algebraic-structural mode because (s)he is capable to interconnect different concepts (s)he has learned at different educational levels. Still, it is unclear whether (s)he derived a conclusion that both equations actually represent parallel lines, because it is not explicitly stated anywhere in the solution (for example because both lines have the same slope equal to 2) and because (s)he did not offer a geometric mode of description on the worksheet either.

In summary, it appears that students posses strong enough visual and symbolic representations of determinants (for example, Figure 5.50, Figure 5.51, Figure 5.54 and Figure 5.55), so that they can now reflect on and connect them to other concepts as lines on planes, besides the previously used concepts as areas of parallelograms

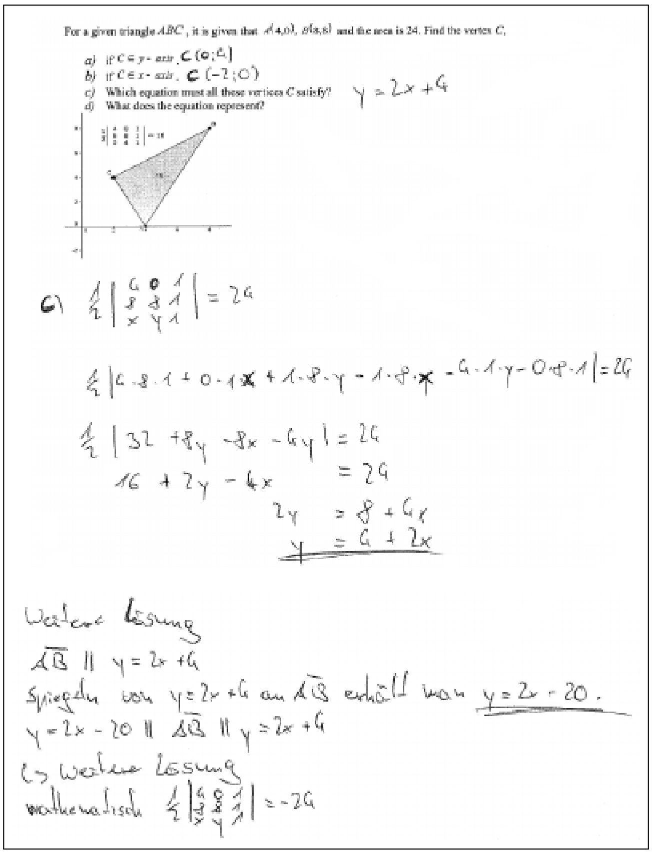


Figure 5.54: Student's 5 Answer to the Task 5

for example. This meets the last guiding feature of conceptual understanding about establishment of external connections between more concepts in problem solving situations (Subsection 2.1.1, p. 37 of this thesis). Finally this relates to investigations on the last *ARQ4*, p. 63. Further analysis on the both ARQ3 and ARQ4 follows through the assessment strategies in the next Subsection.

5.4.5 Assessment for Determinants

Assessment through the Authentic Performance Tasks

Authentic performance tasks do not evaluate manual manipulations or body movements, but are designed to measure higher-order thinking skills, namely the three modes of thinking about vectors, dot product of vectors and determinants. Such an example is the *verbal communication* of the student S6 (lines [53] to [55], p. 124) who connects the area of the parallelogram with the absolute value of the corresponding determinant, thus *successfully* switches from geometric to algebraic mode of thinking about determinants.

A distinction between the degrees of students' understanding is made clear by the oral communication between the students S2 and S4, through the excerpt of the transcript of the discussion (lines [24] to [33], p. 120) about Axiom 3a for determi-

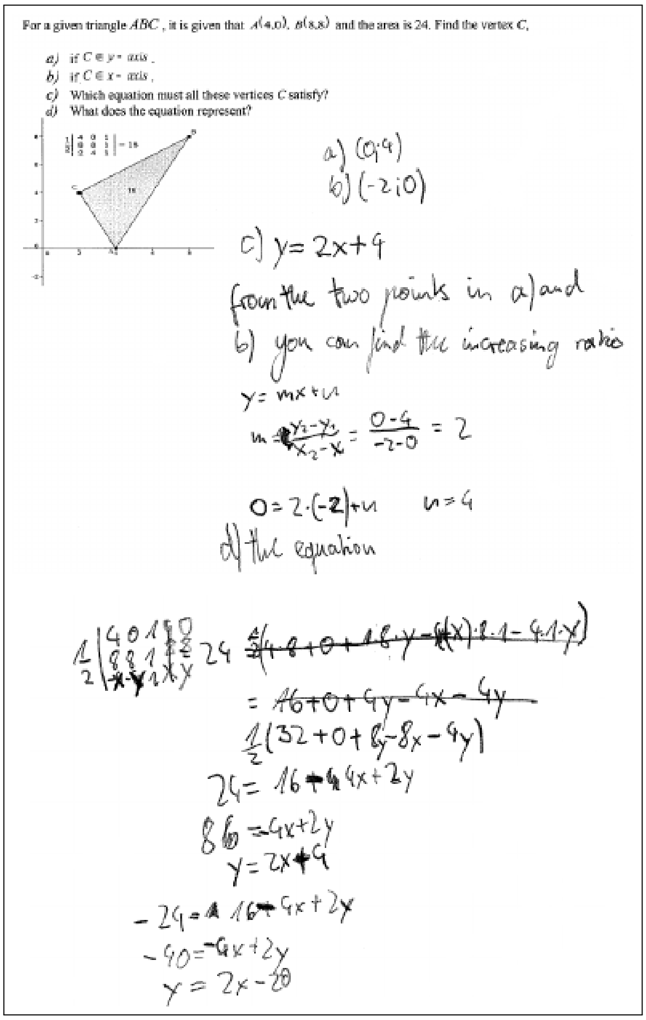


Figure 5.55: Student's 6 Answer to the Task 5

nants. The verbal protocol shows that S2 is *slower* in deriving conclusions than S4, although they both end up with *successful* answers.

Further tracking by the verbal articulations of the student S2 (lines [4], [6], [8], [25], [27], [31] and [33], p. 117) and of the student S3 (lines [12], [14], [34], [36], [38], [40] and [43], p. 118) shows students' learning progress about multi-linearity of determinant functions bases on individual engagements with the interactive applets. These lines show individual cognitive growth of both students, besides their collaboration in the DGE.

Assessment through the Mathematical Journals

Self-assessment through the students' Mathematical journals about determinants is in a similar manner as the one about the dot product (p. 114). Here are some citations of the students' writings.

I understand:

- how to write down determinants, how to calculate with them (addition), connection to parallelograms
- relations between vectors and determinants, determinants (what they are, how to solve them, add them)
- adding determinants and how to solve them, how to draw a 3D cs
- how to write and calculate the determinant
- Die Weise wie man addiert und multipliziert (Translation: The way how to add and multiply)

Knowing the fact that the topic about determinants was completely new, a student reported that (s)he has learned notifications and correct mathematical symbols ("how to write down determinants"), then refers to the algebraic and geometric modes of description. Writings "how to calculate" and "how to solve" show development of procedural knowledge. "Both procedural ability and conceptual understanding are necessary for success in mathematics" ([Porter & Masingila, 2000]; [Hiebert & Carpenter, 1992]). Indeed, not only "how to solve" them, but also "what they are" points out conceptual understanding. The entry "relations between vectors and determinants" is an important one as it points out awareness of the multilinearity property of determinants. The above citations show that four students reported that they understand the additive Axiom 3b for determinants, which was supported by one the applets. One student reported a problem with the geometric visualization of this axiom (see the first entry in the item I do not understand, below). This contrasting feedback of one student may require individualizing instruction ([Borasi & Rose, 1989]).
I do not understand:

- Warum die zwei roten Flächen die blaue Fläche ergeben (Translation: Why the two red areas give the blue one)
- those "parallelograms" but it's OK, I will if we just go on talking about it...
- how determine coordinates of vertices in a coordinate system that is long one unit

The first two students' writings refer to the applet for the additive property of determinants. They served as a resource for an input discussion related to Task. 1. Parallelogram.
Even though students' journals are non-typical for mathematics education, they provided a valuable feedback on students' progress in cognition and their abilities for self-assessment in the created DGE. By writing Mathematical journals, even without use of a rigorous mathematical language, students played active role in assessment (as opposite to conventional perception on students' passive role in assessment,

[[NCTM, 2000], p. 604) by personal engagement in identifying the specific mathematical content, by distinguishing between what they have been instructed and what they have learned (items 1 and 2) and between what they have understood or not (items 3 and 4).

Both alternative assessment strategies used in this study were not kept distant one from another, rather interconnected and integrated in the teaching learning sequence (Section 5.1). Instead of implementing a linear process of assessment (pretest, instruction, post-test), they allowed a non-linear process of instruction and assessment in the same time and context of the DGE ([Young, 1995]). In this way, assessment was undertaken as a continuous process and not just as a product ([Young, 1995]).

5.5 Analysis and Results on Interpersonal, Classroom and Resource Level

In the Sections 5.1 to 5.4 I have analysed the findings of this study on the cognitive level of the multiple data analysis (p. 67). In this section, I continue to analyze the obtained data on the interpersonal, classroom and resource level.

5.5.1 Analysis regarding the ALT

During the experiments undertaken in phases 3, 4 and 5 of the complete cycle of this design-based research, traditional pedagogical media as paper-pencil and board-chalk are supported with digital media. Thus, student-student and student-instructor interactions are not the only ones that appear in the complex environment, but a symbiotic of student-technology and instructor-technology interactions are also included. Technology has the role to mediate and shape different modes of thinking of the concepts in Linear algebra.

Instrumental Orchestrations in the ALT for the Dot Product

The didactical configuration during the ALT for the dot product allows access to the digital equipment, the board and learning materials, which is a base for implementing the collective instrumental genesis. On the beginning of this whole-class teaching unit, a *Link-screen-board* orchestration ([Drijvers et al., 2010]) takes place. This is an instructor centred orchestration which is necessary for the introduction. When the concept of dot product is mentioned, students show sympathy, positive reactions and good working atmosphere (lines [2] to [4], p. 105). Similar positive students' emotions towards this concept have also been detected by [Wittmann, G., 2003], p. 288. Then, a *Discuss-the-screen* orchestration follows (lines [9] to [19], p. 106). This is a student centred orchestration. After that, students have to solve **Task 1**, p. 108 in a paper-pencil manner, thus transfer their applet-based conclusions into conventional mathematics environment (*Link-screen-paper* orchestration). Addition of the written medium for this task in the learning environment aims to support students' individual progress in cognition. The link-screen-paper orchestration was challenging, but meaningful for improving students' competences regarding the dot product.

Instrumental Orchestrations in the ALT for Determinants

On the beginning of the teaching unit about the determinants a short *Explain-the-screen* orchestration ([Drijvers et al., 2010]) is undertaken by the instructor. Using the projector in a computer lab setting (a computer available for a pair of students), the instructor describes the applet. Students are already familiar with the GeoGebra interface, so technical tools and features as sliders and movable points are mentioned, but the focus is primary set on two different modes of description of determinants represented by matching colours (red and blue) on the both applets for determinants. *Discuss-the-screen* instrumental orchestration follows (lines[1] to [68], 117). Afterwards, students have to solve Task 1 to Task 4 in a paper-pencil environment (*Link-screen-paper* orchestration).
The information given above show positive working atmosphere and collective development on a classroom level and may, therefore, be of help for organising further teaching with the designed DGE.

5.5.2 Evaluation on Interpersonal, Classroom and Resource Level

The last item in the students' journals enabled easier trace of students' personal opinions and of the impact of the instructional materials (applets, authentic performance tasks, homework problems). Employing a global measure for evaluation, such as a survey, at the end of all teaching units could not have supplied sufficient information on time.

My personal opinion (overview on today's lesson, teaching method, examples, exercises, homework problems, applications, etc.

- website good way to integrate students and explain the tasks, maybe explain what the website shows in more detail
- I liked the lesson, it's hard to follow the class, because it is in English. As well, I liked that everything explained and mostly with an example.
- good examples and much exercises but much homework
- good, maybe plan more time for answering the questions on the application
- good, maybe explain more in class
- good, but way much homework
- all in all good

- good
- please stop using those Java applets, they don't work anyway! If you want to show us those applets, show it on your computer with a projector.

The statement: "website good way to integrate students ...", proposes sympathy towards the collaborative work of peers when engaging with the applets. This suggests that this student perceives the designed DGE as a medium for interactions not only between the students, but also between them and the created artefact. This student shows wider interest not only for a particular applet, but also for "what the website shows in more detail". This means that the DGE stimulated student's curiosity for further contents or investigations. Student's opportunity to promote her/his motivation and to ask for the availability and access to other information on the website strengthens the value of the suggested DGE on the *recourse level.* "Increased questioning of material" may be attributed to writing Mathematical journals ([Lanigan, 2006], p. 42).
Despite the effort from both the instructor and the teacher additionally supported by a created Dictionary (see Appendix B.), it seems that the English language of instruction may have caused difficulties for this student. This information helps in keeping constant instructor's awareness for students' needs of additional support (in collaboration with the teacher) regarding the language of instruction (an issue discussed in Subsection 6.2.1). Yet, student's writing "[...]mostly with an example" refers to rich *resources* for learning. Further on, the phrase: "[...]everything explained[...]" points out the instructor's role, whereas "I liked the lesson" and "[...]as well I liked that[...]" point out student's personal satisfaction.
Student's writings "good examples", "much exercises" and "much homework" refer to applied teaching/ learning *resourses.* The last student's comment refers to a technical problem which appeared with one of the computers in the lab, probably because of unsuitable Java version which did not supported the GeoGebra applets. After the end of the teaching units one student voluntary created his own applet about the distributive property of vector addition.

Mathematical journals as open-ended assignments ([Adu-Gyamfi et al., 2010]) are time-consuming, seek huge amount of instructor's commitment and are "limited to what they can assess" ([Lanigan, 2006], p. 41). For this reason, they were used as an integral part of a rich pedagogic repertoire for assessment and instruction together with the authentic performance tasks. The applied journals, with effective prompts for short and 'quick' notes (not necessarily extensive prose), seems to have been adequate for acknowledging learning and thinking, capturing the richness of the interactions in the DGE towards conceptual understanding in this design-based research study. A combined use of both assessment instruments and their integration into instruction and learning ([Adair-Hauck et al., 2006]) outlined a deeper insight into the evaluation on four levels of multiple levels data analyses. The journals provided not only sufficient diagnosis on students' cognitive improvements but also efficient feedback regarding the interpersonal, classroom and recourse level. This was the aim of the Phase 6 in the complete cycle of the design-based research (p. 65).

Chapter 6

Concluding Remarks

In this Chapter of the thesis I conduct a retrospective analysis of the overall undertaken research project. I derive and discuss nine conclusions in Section 6.1. Then, I analyse its quality in Section 6.2 mentioning limitations (in Subsection 6.2.1), ways for dissemination and additional contributions of the study (in Subsection 6.2.2). Final recommendations for further design and research follow in Section 6.3. I summarize the contribution of the study in Section 6.4.

6.1 Findings and Discussion

The analysis on the *cognitive level* of the multiple levels data analysis refers to the four *ARQs* which aimed to supply sufficient data regarding the *CRQ* (p. 62).

- The analysis (Section 5.1) regarding *ARQ1* (p. 62) following the HLT (Section 4.1) shows the following findings.
  - The majority of the participating upper high school students possess concept definitions and concept images of *vectors* as classes of arrows which are equal in length, direction and orientation, so in connection to the geometric mode of description and thinking (see Preliminary Study, p. 97).
  - Although there is a dis-balance between the geometric and the algebraic concept definitions of vectors in the participating students, the most of the students were able to recall their knowledge when learning the dot product and determinants, after sufficient initial instruction about vectors as n-tuples (see Sections 5.2 to 5.4).

- The analysis (Section 5.2) on the *ARQ2* (p. 62) following the HLT (Section 4.2) shows that:
  - By the traditional teaching of one or even two concept definitions of the dot product, without emphasis on the existing connections between them, many of the students neither could link the algebraic and the geometric mode of description, nor understood the meaning of the resulting scalar (see transcript [5] to [8], p. 105).
  - By an exposure to the designed DGE a participating student successfully externalized his understanding of the dot product by connecting two concept definitions (see transcript [9] to [19], p. 106).
  - The most of the participating students provided written appropriate solutions or good attempts regarding Task 1. about the axiomatic property for symmetry of the dot product (Subsection 5.2.2).
- The analysis (Section 5.3) on the *ARQ3* (p. 63) following the HLT (Section 4.3) shows that:
  - The majority of the students actively contributed in the discussion related to the Axioms 2 and 3a (see Subsection 5.3.1), Axioms 2 and 3b (see Subsection 5.3.2) and Axiom 1 (see Subsection 5.3.3) for determinants in the designed DGE.
- The analysis (Section 5.4) on the *ARQ4* (p. 63) following the HLT (Section 4.4) shows that:
  - The most of the participating students successfully transferred their knowledge from technology to paper-pencil medium when solving the Tasks 1. Parallelogram, 2. Triangle, 3. Trapezoid and 4. Locus. They created a wealthy of colourful geometric visualizations (Figures 5.22, 5.24, 5.26, 5.37a), 5.38a), 5.39a), 5.40a), 5.43, 5.44, 5.45 and 5.51) and accompanying arithmetic-algebraic solutions (Figures 5.23, 5.25, 5.27, 5.36b), c), 5.37b), c), 5.38b), c), 5.39b), c), 5.40b), c), 5.43, 5.44, 5.45 and 5.51) (see Section 5.4).

A look back at the above answers to the four ARQs allows me to say that, regarding to the *CRQ*, the designed DGE stimulated development of *conceptual understanding* of the dot product and determinants in relation to the *five features* (p. 37) as follows.

1. The most of the participating students are able to distinguish between *what is and what is not* a dot product (transcript [1] to [5], p. 105).
2. The participating students had a chance to learn that *more than one concept definitions* for the dot product exist (see Subsection 5.2.1). There are students who were able to switch from one into another concept definition of the dot product in the DGE (see transcript [9] to [19], p. 106).

3. The most of the students got familiar with the *geometric-, arithmetic-algebraic- and some aspects of the abstract-structural mode of description and thinking* of the dot product (see Section 5.2), i.e determinants (see Sections 5.3 and 5.4).

4. The most of the students could learn *properties which construct the axiomatic definition* for the dot product (see Subsection 5.2.2) and determinants (see above findings for the *ARQ3*).

5. The most of the students could *connect* the both concepts with other concepts (in Arithmetic, Elementary geometry and Trigonometry) and *apply* them in problem solving (see above findings for the *ARQ4*).

In addition to the above short statement for the results regarding the feature 4, I comment the following.
It is possible to learn properties which construct an axiomatic definition without complete awareness of the whole complexity of axiomatic systems at upper secondary education. Although dot product of vectors is a concept in most of the Linear algebra curricula for gymnasiums, properties which construct its axiomatic definition are sometimes omitted, vaguely mentioned or offered as additional, non-compulsory content in school textbooks or instruction (for example, see textbook [Adam et. al., 2007], p. 114, in which they are offered on a CD). This study shows that many of the participating students are able to comprehend properties of an axiomatic definition not only for the dot product (Subsection 5.2.2), but also for determinants (Subsectios 5.3.1 to 5.3.3). Such comprehensions are essential for grasping initial ideas, such as axiomatic-structural aspects, of the general and unifying theory of Linear algebra (see paradigm questions in Subsection 6.3.2).

The analysis on the *interpersonal, classroom and resource level* of the multiple levels data analysis (Section 5.5) showed that the applied Design-Based Research methodology seems as adequate for examining complex didactical phenomena in technology rich classroom settings.
This Design-Based Research does not claim a perfection of its product ([The Design-Based Research Collective, 2003]). An observation of its successfulness depending on the context in which it was created, implemented and evaluated in school settings was carried out instead. It offers a realistic picture of the motivation, design and accomplishment of the innovation. It seems that the chosen methodology suited the proposed model of learning and served to promote answers of the auxiliary and main research questions. Backtrack of the designed instrument ([Cobb et al., 2011]) is an advantage for improvement of the iterative longitudinal design along the trials. In this study, it shows that its final product has undergone minimal modifications during the experiments which may not have essentially affected the initial idea, but have certainly contributed to the upgrade of the artefact.

6.2 Reflections on the Quality of the Study

This section has its focus on the quality of the underlying research, by assuming quality of: (1) the posed research question, (2) the applied research design and (3) the obtained research outcomes ([Niss, 2010]). The discussion on each of these three issues is supplemented by quality criteria according to [Creswell, 2013] and [Sierpinska, 1993].

First, it seems that the central research question (CRQ) in this study: How do students develop conceptual understanding of vectors, dot product of vectors and determinants in a dynamic geometry environment (DGE) at upper secondary education? (p. 62) clearly shapes the *set of problems* for the development of conceptual knowledge in Linear algebra and the support which may be provided by the digital technologies during the transition period from secondary to university education. The CRQ precisely *addresses what* is the content to be researched (conceptual, rather than procedural understanding of vectors, dot product of vectors and determinants), *in which context* (a designed dynamic geometry environment) and the *target group* (upper secondary students). It is a *genuine* and *non-rhetorical question* whose complexity, especially because "conceptual understanding" may be considered as an elusive term, requires auxiliary research questions (ARQs) which would refer to its particular parts. ARQs try to provide answers in accordance to the previously stated guiding features of conceptual understanding (Section 2.1 and particularly Subsection 2.1.1) of each of the Linear algebra concepts in concern. They address concept definitions and concept images, multiple modes of description and thinking as well as their utilization in establishing connections between former and new knowledge and application in problem solving situations in the created DGE. Requirements of integrating existing theories about each of these features, in order answering the CRQ and thus ARQSs shows that all research questions are indeed *researchable.*

Second, backtracking the undertaken design efforts, decisions and processes through its phases contributes in grounding *trustworthiness* and *repeatability* ([Cobb et al., 2011]) which is possible in appreciation to the nature of the applied methodology. Namely, the undertaken Design-Based research (Section 3.3) aims to answer the posed CRQ by creation of a suitable DGE for supporting development of conceptual understanding. It undergoes seven phases of a complete cycle (Subsection 3.3.1) involving a set of theoretical and empirical methods (Subsection 3.3.2) which have the capacity to offer *justifiable* answers to the ARQs and the CRQ. Thus, specific parts of the lengthy process (each of its phase) have defined aims and resulting outcomes which precondition further proceeding of the next phases in order completing the whole cycle (Figure 3.2, p. 65 and Table 3.1, p. 66). Especially, phases 3 to 6 seem to provide strong answers to the corresponding ARQs (as specified in Table 3.1). The whole Design-Based research in this project has a preciously defined *scope* and *range* (see Subsections 6.2.1 and 6.2.2). Its range of applicability could be widened because, by minor adaptations, it could be tested in other contexts, for example with university students. It seems that the undertaken design has the potential to tackle questions, as for example about the capacities of technologies in supporting not only the geometric and arithmetic-algebraic modes of description and thinking, but establishment of their correlation to the abstract axiomatic modes of description and thinking of Linear algebra concepts. Such potential is recognized in the possibility for starting a new cycle of design research as a continuation of the outcome of the last phase. The outputs of each of the seven phases in the cycle may serve as a *control measure* of the background data and investigation factors in this direction. It appears that application of other 'simpler' research methodologies (for example, controlled experiments or case studies) have limited potentials in answering complex RQs as the one in this study, because of its demands as creating, producing, testing, evaluating and disseminating a DGE for supporting students' conceptual understanding in Linear algebra.

Third, the research analysis and results (Chapter 5 and Section 6.1) seem to be *non-trivial*. They are interpreted through different theoretical frameworks as suggested in the background expose of the thesis (Chapters 1 and 2). Namely, they were interpreted through the frame of long-term development of conceptual understanding along more levels of mathematics education including the transition periods. Further on, the growth of conceptual development was discussed in connection to concept definitions and concept images ([Tall & Vinner, 1981]) and to multiple modes of description and thinking of Linear algebra concepts ([Hillel, 2000]; [Sierpinska, 2000]). Additionally, the theories about instrumental genesis in a whole class technology based learning circumstances ([Artigue, 2002], [Drijvers et al., 2010]; 2013) served the multiple levels data analysis. Incorporation of all these different theories shows that the findings in this study correspond to findings reported in existing literature. Interpretations of the findings are accompanied with the data as excerpts of the transcript video recordings or students' writings and there are no contradictory claims among them, which sustains the *validity* of the study. The findings of this study suggest that axiomatic approaches deserve place in upper high school mathematics under conditions that they are intercepted in cohesion with primary geometric and algebraic approaches and digestible for the students' cognitive maturity with additional aid of technological tools. Such findings confront previously applied and abandoned axiomatic approaches in high school during the "New Math" era. These considerations point out that this study has a "*theoretical relevance*" because it "broadens and deepens our understanding on teaching and learning phenomena" ([Sierpinska, 1993], p. 38) of Linear algebra concepts in a DGE on the basis of existing theoretical frameworks. In particular, it contributes to our understanding of possible ways for implementing axiomatization in upper high school settings, and in this sense it may be that it even extents current known research. The "*pragmatic relevance*" ([Sierpinska, 1993], p. 38) is explained in Section 6.3. The results in the study are presented through the Pre-study (Section 5.1) and three teaching units (Sections 5.2 to 5.4), such that each of these parts directly addresses and tries to answer a particular ARQ. The *range of the findings* may even be widened by posing different RQs as for example, what is the role of the instructors in supporting students' conceptual understanding in the designed DGE. The last phase of the research, dissemination of the final product of the study (Subsection 6.2.2), promotes authorized open-source interactive materials as part of the designed DGE which sustains the *originality* of the undertaken research study. It offers a possibility for further public sharing of the designed teaching and learning instrument, and for opening a debate in the research community or among didactical engineers, designers, authors of text-books and even curricula developers, thus promotes wide practical and theoretical applications and relevance. This broad spectrum of potential users of the materials and findings reported in this study demonstrates that they could be used in other contexts on national and international level which strengthens the *reproducibility* and *significance* of the study in complete.

6.2.1 Limitations of the Study

This part points out four limitations of this research study.

The first limitation is that the study is a *short-time* one.

Evidence on a long-term development of conceptual understanding from upper high school to university level of Linear algebra may also be collected by long-lasting studies. In the constrains of this study it was only possible to derive some assumptions, though not claims, for the students' future knowledge development on the basis of the three modes of description and thinking of the dot product and determinants. Yet, such assumptions are helpful for further investigations on a long term.

The second limitation is having the *researcher in the role of instructor*.

The instructor primary facilitated students' instrumental genesis and not her own, by resisting giving immediate answers of students' questions in any occasion when possible (verified by the transcripts of the recordings), by stimulating collaborative work and interactions in the technology-rich classroom (asking open questions as: what do you think, can you explain, what do you see on the applet etc.), and by securing students' autonomous individual written works (in Pre-study, teaching units and assessment).
The complexity of chosen research methodology consisted of seven phases of design research practically preconditioned this decision. Compared to other qualitative research methodologies such as case studies or analysis of one to one interviews for example, the chosen methodology has many advantages. Namely, it tries to include as many as possible students participating in the experiments instead of a couple

of students (often not more than three students) chosen by the interviewer or the researcher according to his/her own criteria. Namely, the researcher tried to instruct the students and present their learning outcomes without any prejudices. In the later discussions with the teacher it was discovered that even less achieving students actively participated in the ALT (e.g. see the progress of S2 in the transcript for determines, p. 5.3.1). The question whether the selected sample of students is representative and the risk of direct researcher's influence on an individual student's outcome, which are typical for the mentioned methodologies, seem to have been avoided in the chosen one. The biggest importance of this methodology is that students are in their own natural environment, in a classroom surrounded with their classmates, with whom they interact and exchange ideas, while in the other mentioned methods this socio-cultural context is completely omitted. The presence of the researcher in this environment appears not to have pressured student's work and results because of the invested researcher's effort trying not to affect students' outcomes as much as possible. Interpretations, reports and comments of outcomes and assessment along the whole research use mask names of cooperators (Teaching assistant 1, Teaching assistant 2, Tutor 1, Tutor 2, etc.) due to ethical reasons and mask names of students (Student 1, Student 2, etc.) due to sensitiveness of adolescence issues ([Creswell, 2013]).
Teachers may not be confident in working in a technology-rich classroom, whereas researcher's familiarity with the epistemic values of the exploited tools in the designed DGE may affect decisions on a particular instrumental orchestration and a didactical performance ([Drijvers et al., 2010]). Tight time schedule for the undertaken experiments and teacher's focus on topics important for an external testing (Abitur), which does not include determinants, may also have influenced some decisions.

The third limitation of the study is due to the *language of instruction.*

Mathematics instruction in a foreign language is a very risky action. However, many preliminary steps were undertaken in order avoiding possible problems. Researcher is experienced in mathematics instruction in English as a foreign language. On one hand side, researcher's dedication not only by preparing the Mathematics Dictionary (Appendix B), but also during the didactically performance as an instructor in the actual teaching trajectory, supplemented with the class-teacher's presence in the classroom, both to facilitate, when necessary to translate communication, contributed a lot for the success of the whole experimental process. Excerpts of the transcripts of the video recordings report on students' verbalizing problems, sometimes ending up with answers consisted of a single word. Thinking processes might have remained unarticulated or resulted only with a final conclusion. Missing English vocabulary is evident in excerpts [10] and [12] about the dot product, p. 107 and in the lines from [61] to [68] about the learning of Axiom 1 for determinants, p. 126, for example. On the other hand side, mathematics instruction in English sounded very challenging for the students. They demonstrated their high performance and skills in English as if this non-everyday situation was an additional motivation for their devoted participation. Anyhow, it certainly could not be claimed, but it could at least be assumed that students' achievements could have been even better if the language of instruction was their mother tongue German instead of English.

The fourth limitation refers the *selection of the questions, tasks and problems.*

Designed materials involving applets, static visualizations, guiding questions, tasks and assignments aimed establishing unfamiliar structures of already familiar concepts and modes for the students (when they are separately treated), as for example establishing a link between the volume of the unit cube and the determinant of the unit 3 by 3 matrix. The applied particular choice also aimed basic introduction to formal definitions based on visualizing particular definition axioms. Deductive proves based on such formal definitions and sophisticated axiomatic systems were not part of the analysis in the study. Another choice of appropriate set of tasks in this direction and applicable in the designed environment is certainly possible.

6.2.2 Dissemination and Additional Contributions of the Study

The final Phase 7 of the complete cycle of design-based research (Figure 3.2, p. 65 and Table 3.1) involves dissemination and impact of its product. Outcome 7 in the cycle is a *Small-scale dissemination* which was undertaken trough the Actual Learning Trajectory (ALT) including assessment and measurement of its impact in classroom settings (Chapter 5). Moreover, the goal is not exclusive individuals' assessment, but also evaluation of many other ongoing activities in the dynamic environment. For example, feedback on the resource level of the multiple-levels data analysis: availability and access to the learning materials and their integration in the teaching and learning activities (Subsection 3.3.2), is also considered relevant. Features of the DGE which are invariant all the way through the experiments, as for example constant focus on supporting multiple modes of description and thinking in Linear algebra, are therefore meaningful for evaluation and further adaptations accord different needs. Simultaneously, *large-scale dissemination* involved spreading the designed instrument through internet. Thus, complete designed instrument (artefacts, Variotional Dragging Schemes (VDSs, p. 91 and p. 82) and techniques), including applets, related dynamic worksheets with guiding questions, tasks and additional information (as given in the HLT in Chapter 4) were uploaded on the open-source portal GeoGebraTube, (see Web Resources [4], p. 208). Primarily each of the applets has been uploaded on the portal, but finally all materials were shared in a form of a GeoGebraBook (Appendix A in this thesis).
This means that the Outcome 7 of the last phase completes the whole cycle of design-based research and serves as an input for further research and design (see

Section 6.3). The design is reusable and adaptive also for university level, following the remarks for example in notation as suggested in the axioms in Section 4.2. Adaptations of the applets may be favourable for changing the sliders, vectors' components and determinants' entries into real numbers instead of integers, as it is done on the Applet 2 for the additive property of dot product of vectors in Section 4.4 (or Appendix A). Besides spreading the design on internet, the dissemination and impact also include conference contributions and publications during the project (see [Filler & Donevska-Todorova, 2012]; [Donevska-Todorova, 2010]; 2011; 2012a; 2012b; 2014; 2015; 2016a; 2016b).

Recommendations for Teaching with the Designed DGE

The created GeoGebraBook (Appendix A) for Linear algebra available on GeoGebraTube seems to be suitable material for teaching and learning. On its first side it contains a short explanation about the aim of the book, information about the target group, thus students at the age of 15 to 18, and the language, which is English. It would be recommended to potential users of the book that they previously get familiar with the hypothetical teaching and learning trajectory, especially with the Variational Dragging Schemes (VDSs, p. 91 and p. 82 in this thesis). In order avoiding misconceptions, information about existing relationships and differences between concepts, for example matrices and determinants, should be comprehensive and descriptive ([Aygor & Ozdag, 2012]) as they have been addressed for example in Subsection 3.1.3. Therefore applets contained in each of the topics of the book are accompanied with the discussing and guiding questions, (see for example the last two pages in the GeoGebraBook in Appendix A) which are discussed in Chapter 4 of the thesis, acting together as interactive worksheets. From teachers' point of view, it is important that decisions about appropriate use of particular media (table, paper, technology, etc.) happens at appropriate exact part of the instruction process. In this sense, instrumental orchestrations suggested in this thesis (actual teaching and learning trajectory in Chapter 5) may be valuable. Soon[1], teachers will also have the possibility to organize and share their teaching processes regarding the GeoGebraBook within a group of users, for example students in one class or mathematics teachers in one or more schools.

Dissemination regarding the teaching and learning of Vector Space in the designed DGE

From a broader viewpoint elaborations in the Pre-study and the teaching units may have additional contribution to analysis of early introduction of *Vector Spaces* in school. For this reason, this part argues that students' understanding of linear

[1]The function of Groups is already pronounced by the GeoGebra Team ([Kimeswenger & Hohenwarter, 2014]).

combinations of vectors (S10, S16 and in particular S6 and S12: "vectors are linear combinations one from another" in Table 3.3, Subsection 5.1.1, in other words, are linear dependant vectors) and their oral communication about the homogeneity property, thus Axiom 3a with the use of the Applet 1, for example in the excerpt of the transcript [9] to [23] are in contribution towards development of the concept of a Vector Space. Although Vector Spaces are not in the focus of this research, this section offers some remarks which may be important for raising the axiomatic-structural thinking. For example, discussing questions about vectors $\vec{u}$ and $e\vec{u}$ (or $\vec{v}$ and $k\vec{v}$) and their interpretations with sliders in the designed DGE (Applet 1) can be brought into context of the *closure property under scalar multiplication*:

$$\vec{u} \in V,\ k \in K \text{ then } k\vec{u} \in V$$

and Axiom 1 of the designed axiomatic approach can be brought into context of the *multiplicative identity axiom* of Vector Spaces,

$$1 \cdot \vec{u} = \vec{u},\ \forall \vec{u} \in V$$

In a similar way, discussing questions about additive Axiom 3b with the use of the Applet 2 in the designed axiomatic approach for determinants can be connected with the *closure additive axiom* of Vector Spaces

$$\vec{u}, \vec{v} \in V, \text{ then } \vec{u} + \vec{v} \in V,\ \forall \vec{u}, \vec{v} \in V$$

Neutral additive element and *Commutative property of vector addition* can also be investigated with the use of Applet 2 or with supplementary applet (see Appendix A in this thesis and [Filler & Donevska-Todorova, 2012]). This research project also offers an applet for the *associative property of vector addition* (see Appendix A in this thesis and [Filler & Donevska-Todorova, 2012]). Axioms: homogeneity Axiom 3a and additive Axiom 3b through the Applets 1 and 2 serve for development of multi-linearity of determinants. Moreover, they may be viewed as contributions towards connecting them with *distributive axioms* of vector sums and scalar sums axioms of Vector Spaces. These are only brief considerations which support transition from isolated pure geometric or pure arithmetic modes of descriptions and thinking about vectors, typical for high school, towards the algebraic-structural modes of descriptions and thinking about Vector Spaces at university level of education. The existence of interconnections between Linear algebra concepts offers many possibilities for embedding all three modes of description and thinking into coherent content with help of technology. These remarks address designers and teachers to consider additional tasks, appropriate language and analytic-structural mode of thinking for further feasible implementation of the suggested instrument into practice (see Section 6.3).

Further dissemination includes sharing of dynamic applets for:

- Commutative property of vector addition (Appendix A and [Filler & Donevska-Todorova, 2012])
- Associative property of vector addition (Appendix A and [Filler & Donevska-Todorova, 2012])
- Additive property of dot product of vectors (Applet 2, Figure 4.5 , p. 80 and Appendix A)
- Solving a system of two linear equations with two unknowns with vectors (GeoGebra Tube [4], in the Web Resources, p. 208).

These applets were not part of the undertaken actual learning trajectory (ALT), but are publicly shared through publications and conference contributed talks, and available on GeoGebra Tube.

Impact of the study on the Macedonian practice

While Vector spaces are part of the Macedonian curriculum for upper high school (gymnasium) Linear algebra and Analytic geometry, determinants are still introduced as tools for solving systems of linear equations (SLE) on the beginning of the course within the same curriculum frame. Such combination of an abstract axiomatic approach for Vector spaces and pure arithmetic approach for concepts as determinants seems a bit incoherent within one content domain. The practice of treating determinants trough SLE appears to have been aged (see epistemological analysis of the historical development in Chapter 1 of this thesis) and abandoned in the west European countries due to the use of Gaussian eliminations for such purposes. Gaussian method is also part of the Macedonian curriculum for upper high school Linear algebra, thus a revision or better balance of the curriculum content for Linear algebra in gymnasiums would be preferable.

6.3 Recommendations for Further Design and Research

This section points out some recommendations for potential usage of the created artefact and instruments in further design (in 6.3.1) and further research (in 6.3.2).

6.3.1 Recommendations for Further Design

Suggestions for further design concentrate on eventual adaptations of the design for other contexts or on widening the existing content of this design.

1. Adaptations of the design for other contexts

> Sustainable innovation requires understanding how and why an innovation works within a setting over time and across settings ([The Design-Based Research Collective, 2003], p. 6).

The suggested design in this doctoral project may be applied in the context of an introductory university course Linear algebra. Such desire would certainly require particular adaptations of parts of the design, but keeping its main idea about deepening conceptual understanding by integration of multiple modes of description and thinking consistent. What may be needed in the university Linear algebra context may be interventions on the applets as setting real numbers (instead of integers) as scalars (sliders), vector components and determinants entries. This remark may also be considered relevant for the teaching practices at university level responding to emerging features of this kind of a setting ([The Design-Based Research Collective, 2003]) and the need of increased level of sophistication and generalization. Additionally, a set of new tasks in direction of axiomatic-based proving would be recommended. In this way, "the learning community extends not only horizontally across a classroom, but also vertically" ([Collins, Joseph, & Bielaczyc, 2004], p. 22) across more levels of education.

2. Recommendations for design of new artefacts in the same DGE

   This part proposes a design of interactive three-dimensional dynamic applets (similar as the static visualization on the Figure in this thesis) which will again base on the same idea for integrating geometric, algebraic and axiomatic-structural modes of description and thinking of Linear algebra concepts. Specific applets may refer to dot product of vectors in 3D, cross product in 3D and determinants in 3D as a natural generalization of the corresponding applets offered in this project.

Both of the above suggestions for further design are possible due to the designers' friendly interface of the DGE. Designed instrument can be approached on-line or it can be first freely downloaded from GeoGebraTube in order adaptations or interventions to be applied. Thus, the design is *transparent* and in contribution to *community building* by sharing. The above information are considered adequate, following van der Akker's 'design principles', to provide a chance for reducing uncertainties and solid ground for undertaking particular choices in further design ([Van der Akker, 1999]). Finally, the design in this project, being theoretically supported and empirically tested, may initiate design of other similar instruments which may be brought in the same context for teaching and learning Linear algebra concepts. In this way, the completed cycle of seven phases in this research study may get a spiral characteristic through a systematic upgrade of the offered concepts serving as prototypes.

6.3.2 Suggestions for Research

The closing section in this thesis represents backtracking of the epistemological analysis of the historical development of the theory of Linear algebra and its didactics undertaken in Chapter 1. It reflects on the paradigmatic issue[2] and historical dilemma about the use of axiomatic approaches in high school. It raises several paradigm questions. Is it time to reconsider axiomatic, thus deductive besides the usual inductive, approaches in high school Linear algebra which were intensively launched and fast spread in the 60s, but then abandoned in the 90s of the previous century? What kind of information and support do curricula developers need to reconsider the abstract phenomena, such as generalizing and unifying character of Linear algebra in upper high school curricula? Have we given up of challenging upper high school students too soon, even underestimated their capabilities, by not offering advanced enough concepts (determinants, vector spaces, subspaces, etc.) or advanced enough modes of description and thinking (axiomatic mode for dot and vector products, for example)? What researchers need are proposals for solutions of the existing problems, instead of complains about how difficult the understanding of some mathematical concepts are for the students. Such considerations may initially sound enthusiastic though, but efforts may be worth. It seems that in this era of a variety of technological tools researchers in mathematics education have not yet exploited all sources. The undertaken study and its results, show that the axiomatic definitions in upper high school deserve a treatment between creativity and formalism. An approach which would consider students' current cognitive status at upper high school, in an innovative way which substantially differs from those suggested within the "New Math" from the 1960s, (which failed partly due to ignorance of geometry, and students' exposure to the enactive and embodied world of mathematics), seems achievable. The project sees a shift in the learning trajectory of conceptually advanced topic of determinants from university to upper-secondary level of education. Early students' engagement in axiomatic approaches, traditionally perceived as university approaches may now be accessible with the use of DGS. This is a central facet derived from this research study, taking into account today's curriculum for Linear algebra and Analytic geometry at the upper-secondary education, which may face certain improvements due to the technological trends in the future. After all, students learn what they are offered. If the dish is not served on the table, how can we expect it to be eaten, or at least tasted?

6.4 Summary

Looking back at the challenging questions which were stated in the Preface of this thesis, I may summarize the following.

By a qualitative analysis of two data sets, I have identified and, therefore, offered *theoretical knowledge about two students' difficulties* in the learning of Linear algebra with the: (i) bi-linearity and multi-linearity (homogeneity and additive) properties of the axiomatic definitions of the dot product and the determinants, respectively

[2]Raising paradigmatic issues is considered "the first criterion for qualifying a study among the leading in the field" ([Drijvers, 2012]).

and (ii) geometric interpretations of their resulting scalars (see Observations and Identification of the Research Problem, p. 56 and p. 61). Moreover, the study shows a way for a possible intervention in a DGE in the following sense.

The study has undergone seven phases of a complete *design-based research* cycle (see Section 3.3) and ended up with a designed DGE (an artefact and instruments) for mediating the understanding of the three modes of description and thinking for the dot product and determinants in upper high school students (see Design in Chapter 4).

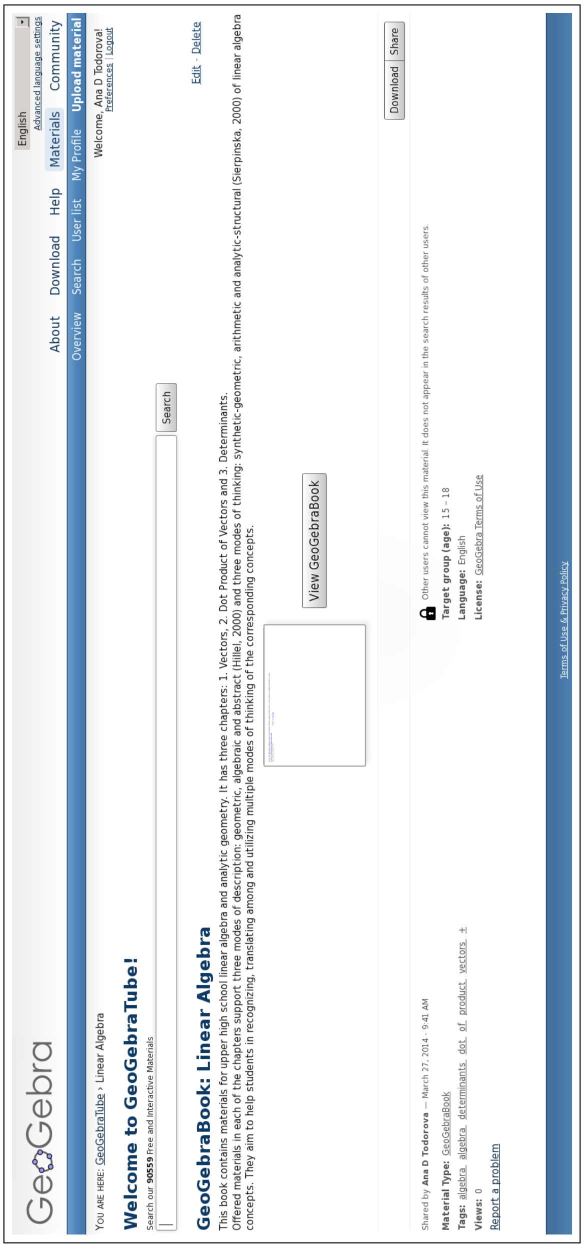
I have found out that traditional teaching of a new concept with a single definition based on one mode of description, or even with two definitions based on two appropriate modes of description, without pointing out the existing connections between them, leads to development of students' poor concept images and weak conceptual understanding (see Subsection 5.2.1 for the dot product). By empirically testing the created DGE, I have come up with findings which show a possible teaching way for supporting the integration not only of the *geometric* and the *algebraic* modes, but also of the *axiomatic-structural mode of description and thinking* of the dot product (see Analysis and Results regarding ARQ2 in Chapter 5.2) and of determinants (see Analysis and Results regarding ARQ3 in Chapter 5.3). Further on, I have obtained successful results also by empirically testing the created DGE for applying such an integration of the modes in problem solving situations towards deepening conceptual understanding (see Analysis and Results regarding ARQ4 in Chapter 5.4). In addition, the study shows a way for assessing the students' development of conceptual understanding of the two concepts (see Subsections 5.2.3 and 5.4.5). Therefore, the five guiding features which were stated in Subsection 2.1.1 may serve for improving the *theories about conceptual understanding* on the basis of concept definitions, concept images [Tall & Vinner, 1981], multiple modes of description and thinking ([Hillel, 2000] and [Sierpinska, 2000]) and their applications in Linear algebra (see Central Research Question in Section 3.2).

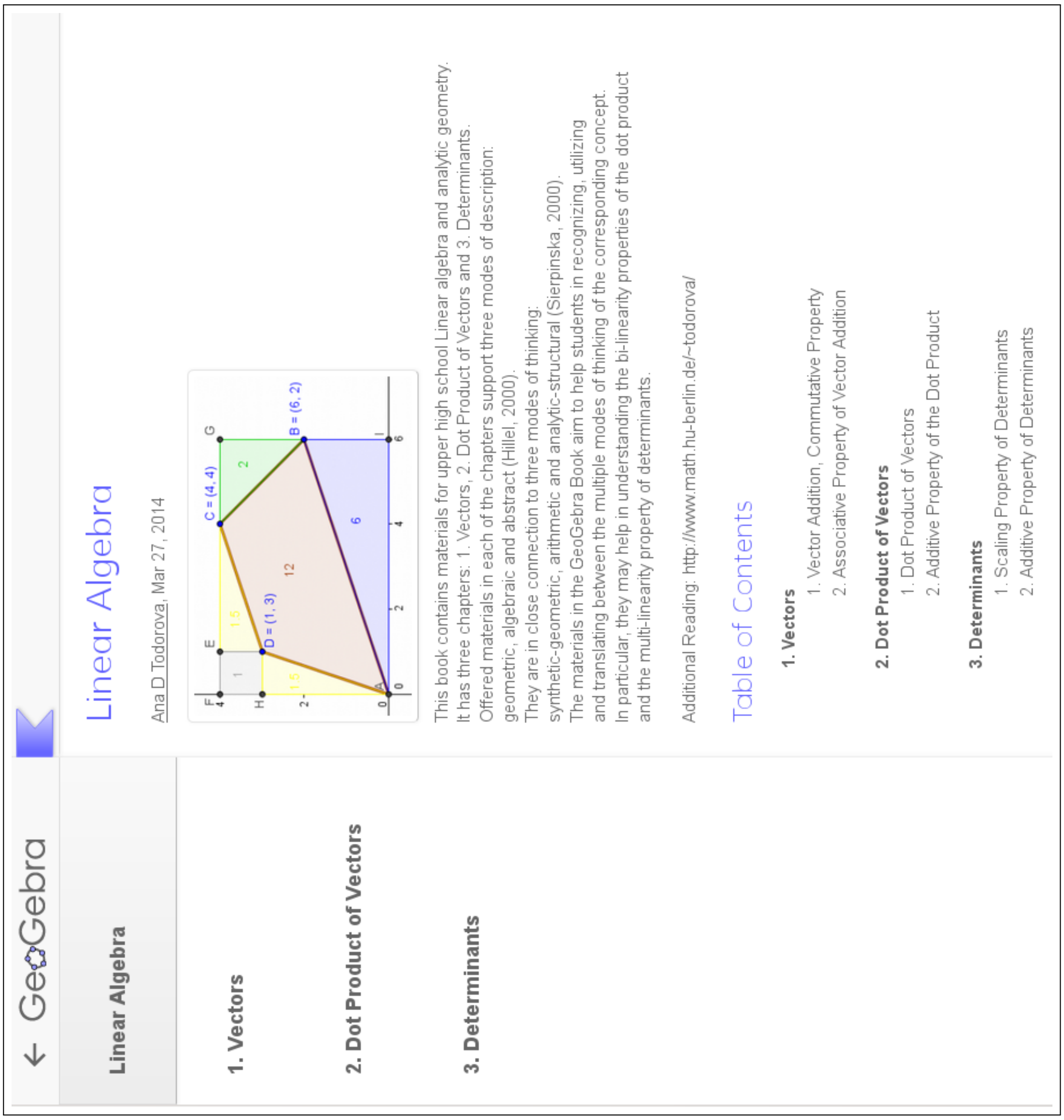
Finally, a retrospective on the whole design-based research study shows that it offers *additional theoretical contribution* for further long-term investigations and discussing paradigmatic questions from a historical point of view, regarding local axiomatic approaches ([Freudenthal, 1971]) in the transition between the secondary and tertiary level Linear algebra which may be facilitated by the use of technology (see Suggestions for Research 6.3.2).

Appendix A

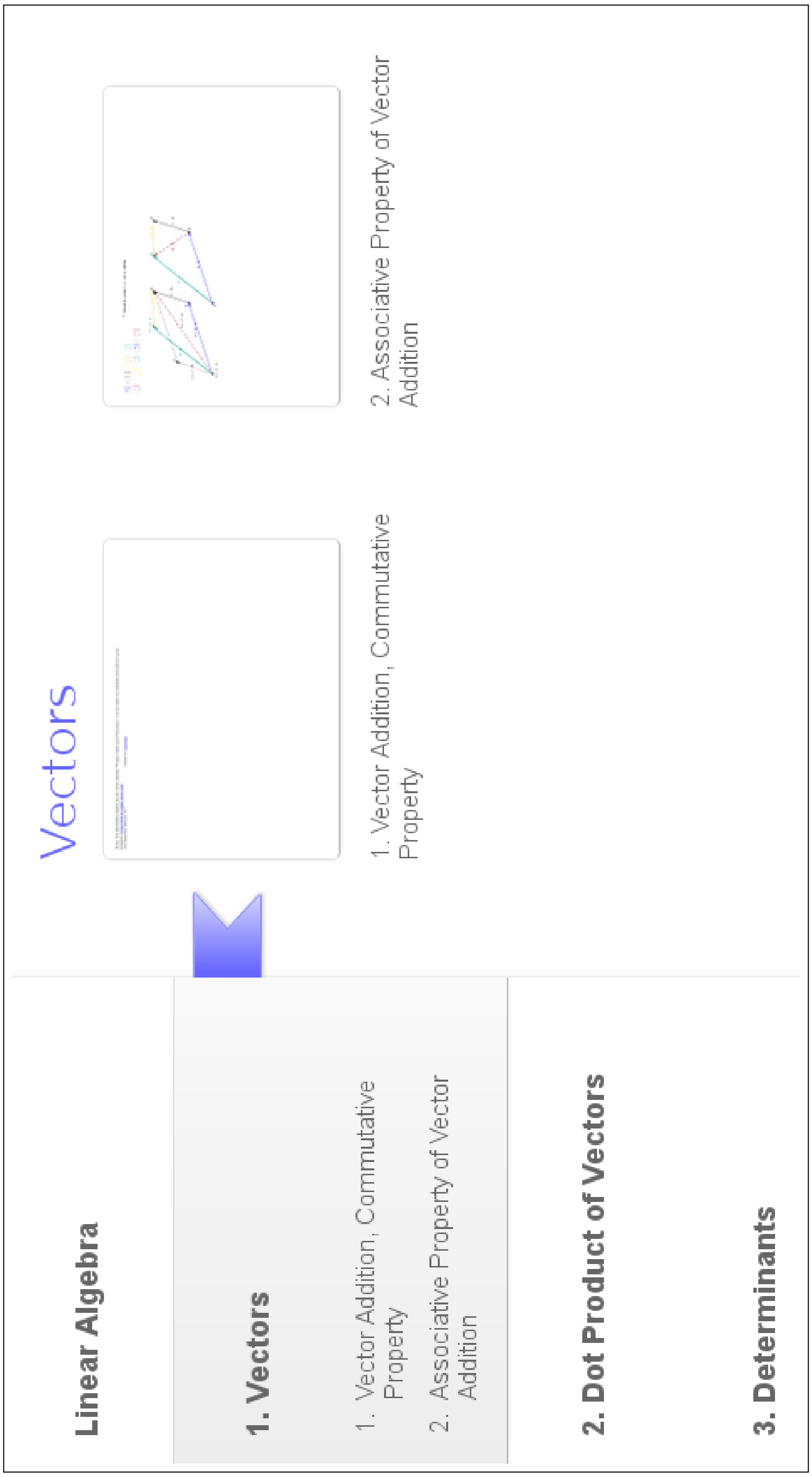
GeoGebraBook

A.1 GeoGebraBook on GeoGebraTube

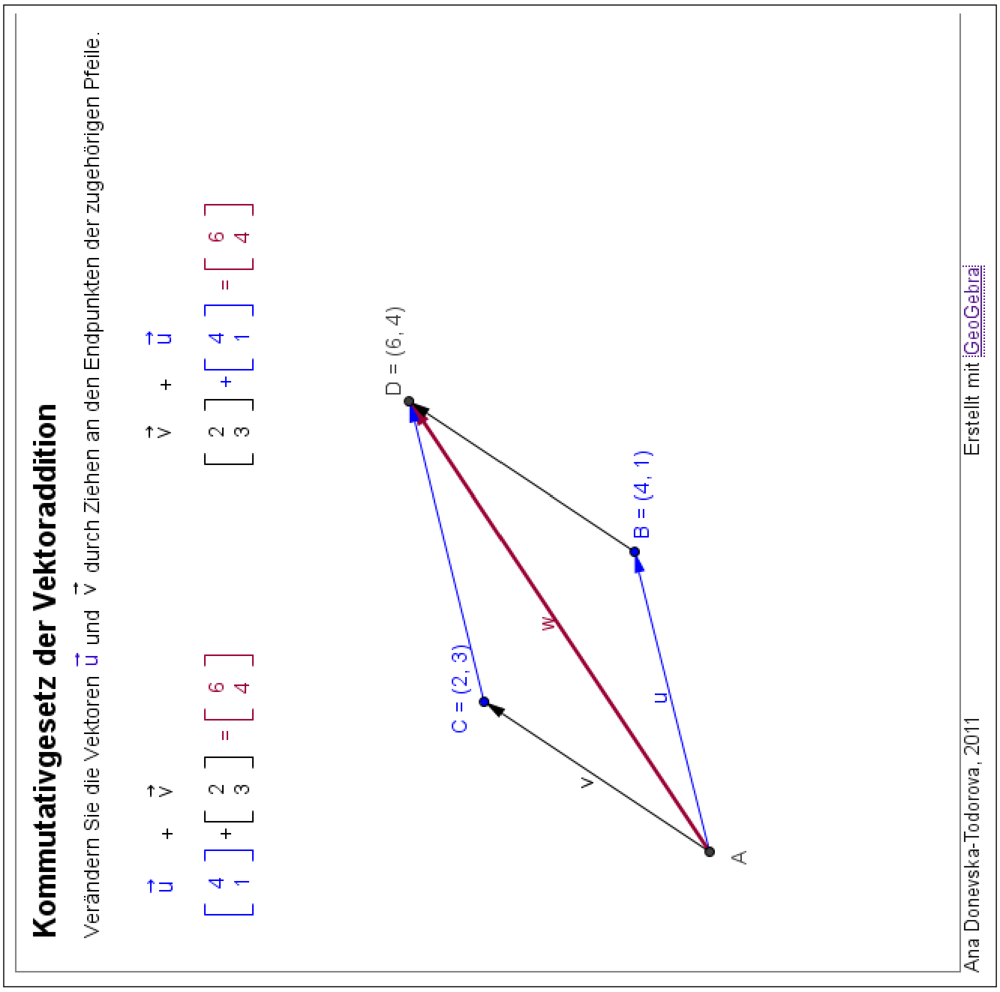



GeoGebra
Linear Algebra
1. Vectors
2. Dot Product of Vectors
3. Determinants
Linear Algebra
Ana D Todorova, Mar 27, 2014
C = (4, 4)
B = (6, 2)
D = (1, 3)
This book contains materials for upper high school Linear algebra and analytic geometry.
It has three chapters: 1. Vectors, 2. Dot Product of Vectors and 3. Determinants.
Offered materials in each of the chapters support three modes of description:
geometric, algebraic and abstract (Hillel, 2000).
They are in close connection to three modes of thinking:
synthetic-geometric, arithmetic and analytic-structural (Sierpinska, 2000).
The materials in the GeoGebra Book aim to help students in recognizing, utilizing
and translating between the multiple modes of thinking of the corresponding concept.
In particular, they may help in understanding the bi-linearity properties of the dot product
and the multi-linearity property of determinants.
Additional Reading: http://www.math.hu-berlin.de/~todorova/
Table of Contents
1. Vectors
1. Vector Addition, Commutative Property
2. Associative Property of Vector Addition
2. Dot Product of Vectors
1. Dot Product of Vectors
2. Additive Property of the Dot Product
3. Determinants
1. Scaling Property of Determinants
2. Additive Property of Determinants

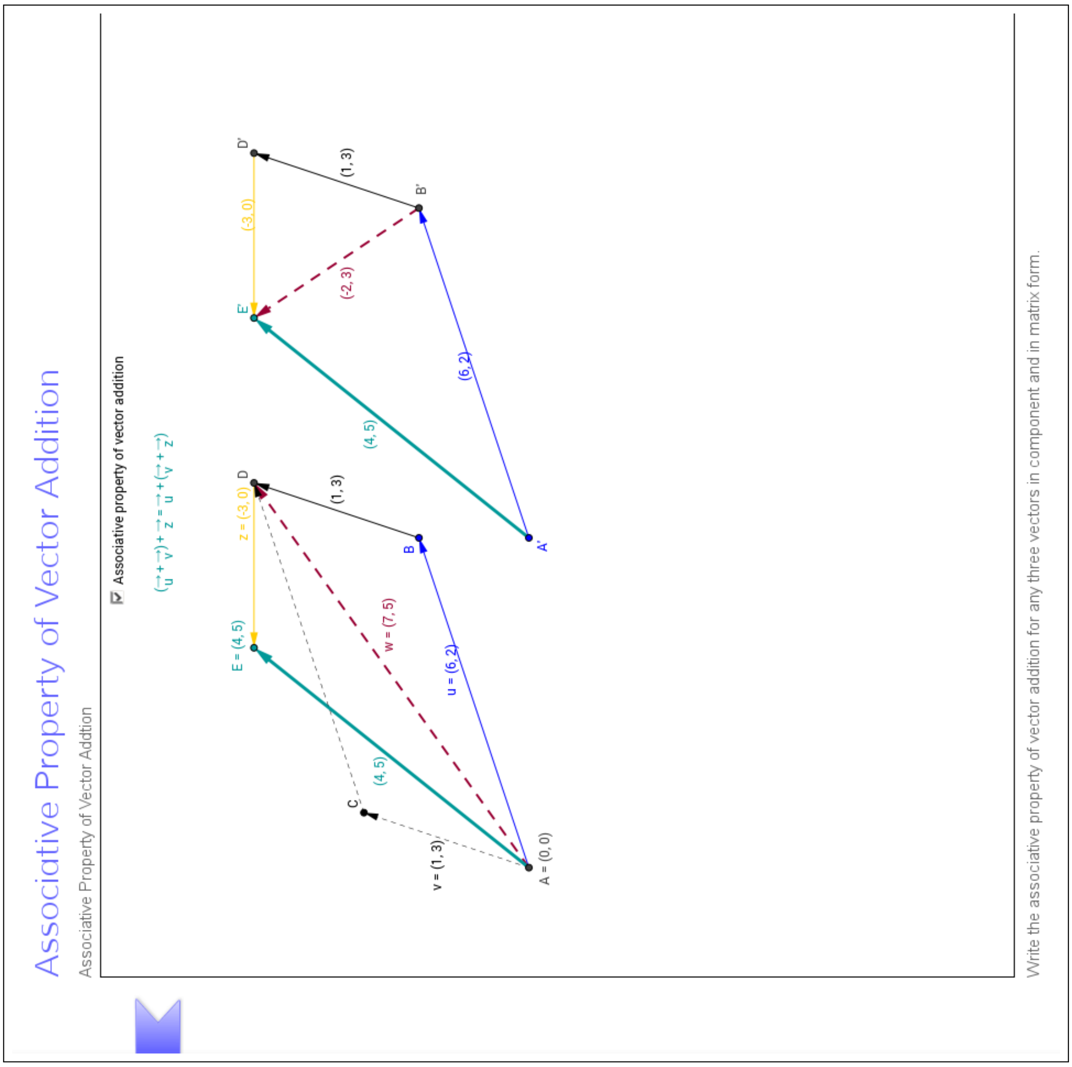
A.2 Chapter 1. Vectors



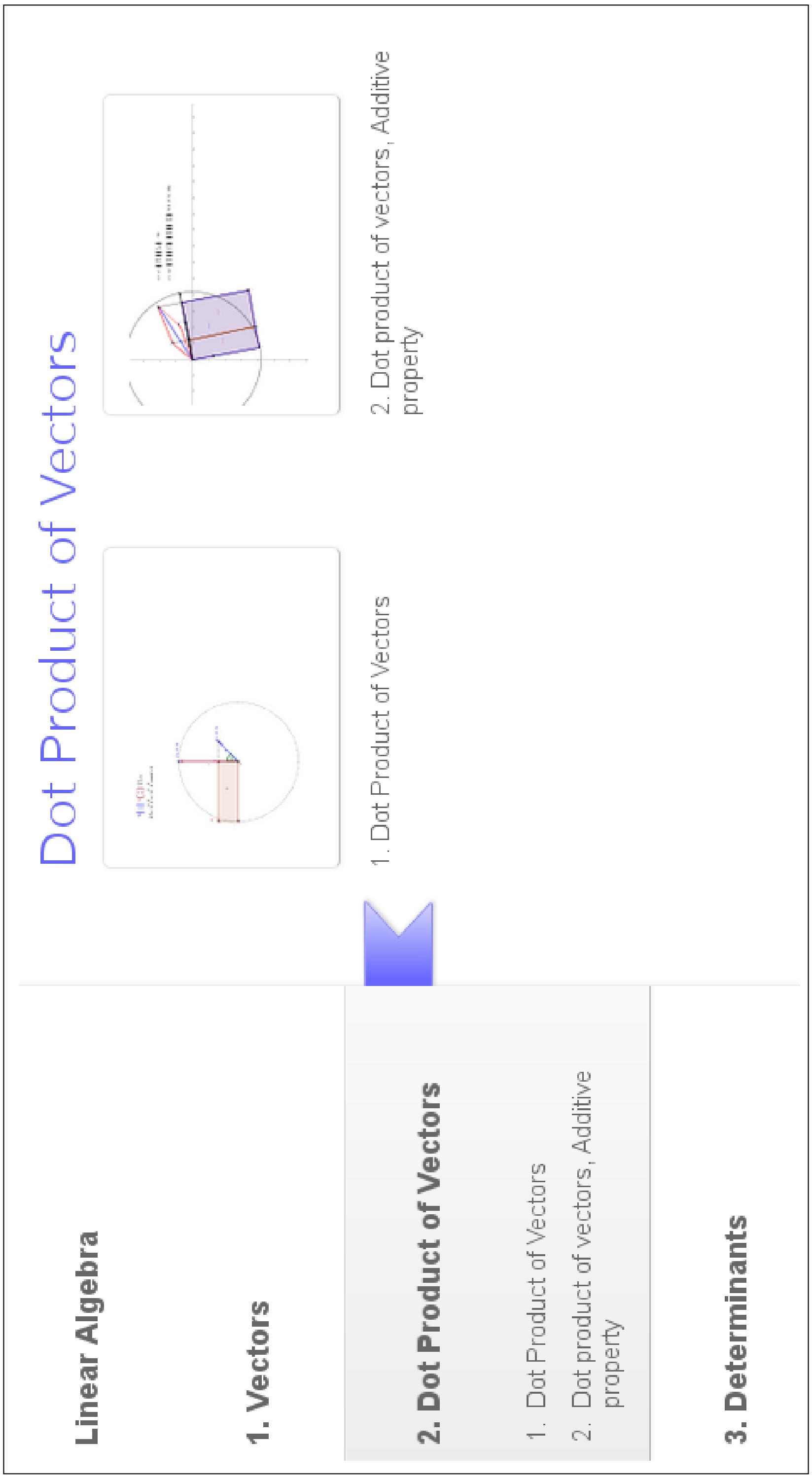
A.3 Commutative Property of Vector Addition



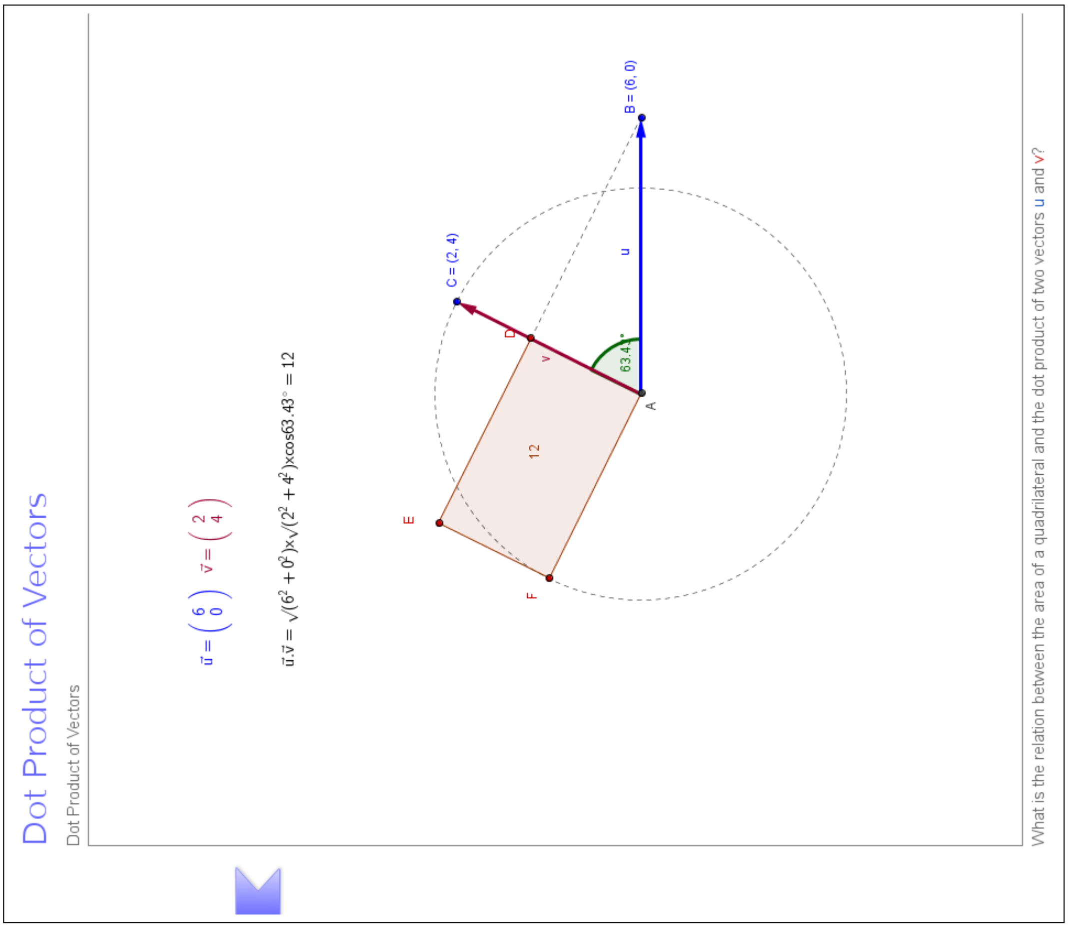
A.4 Associative Property of Vector Addition



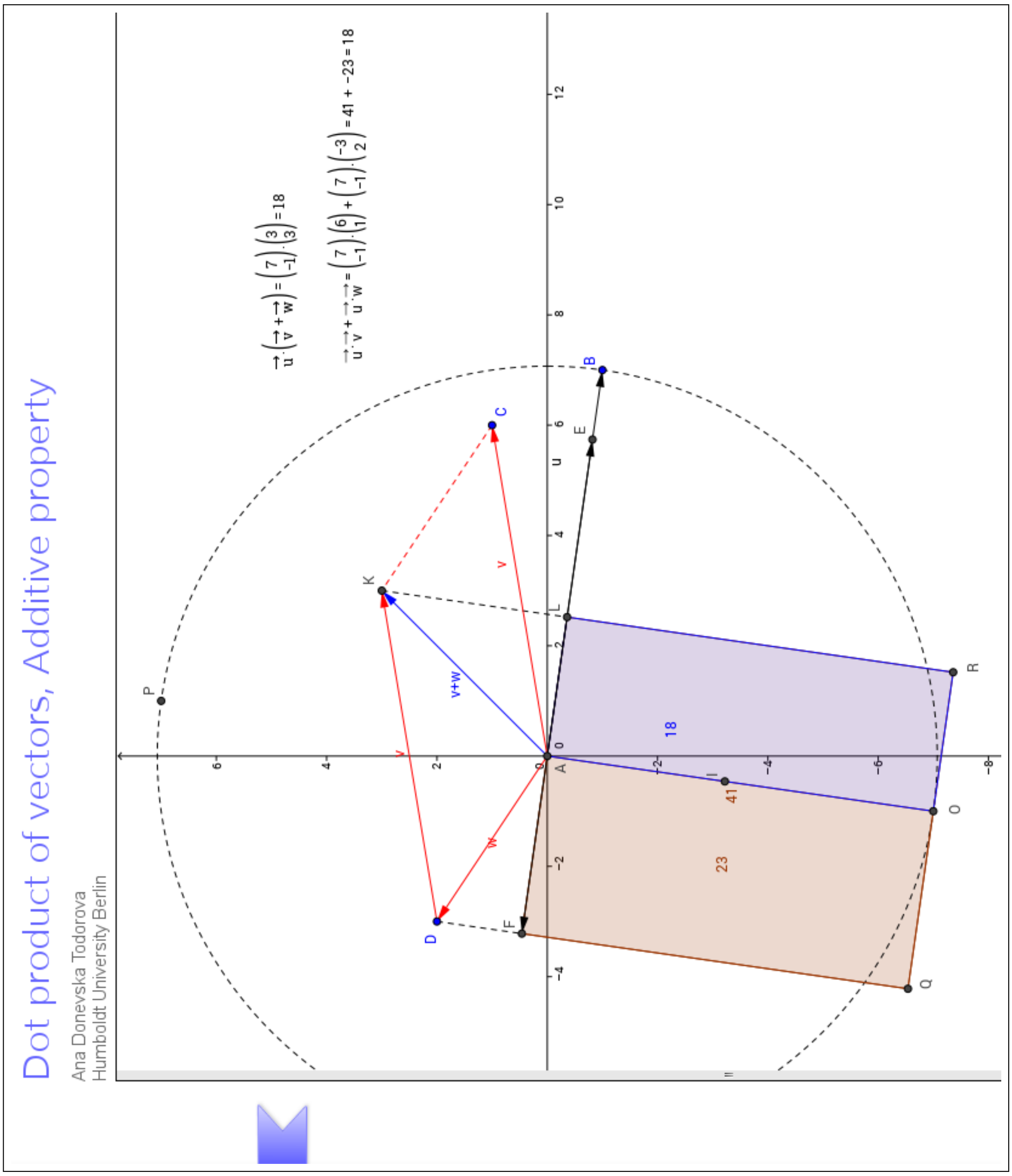
A.5 Chapter 2. Dot Product of Vectors



A.6 Dot Product of Vectors

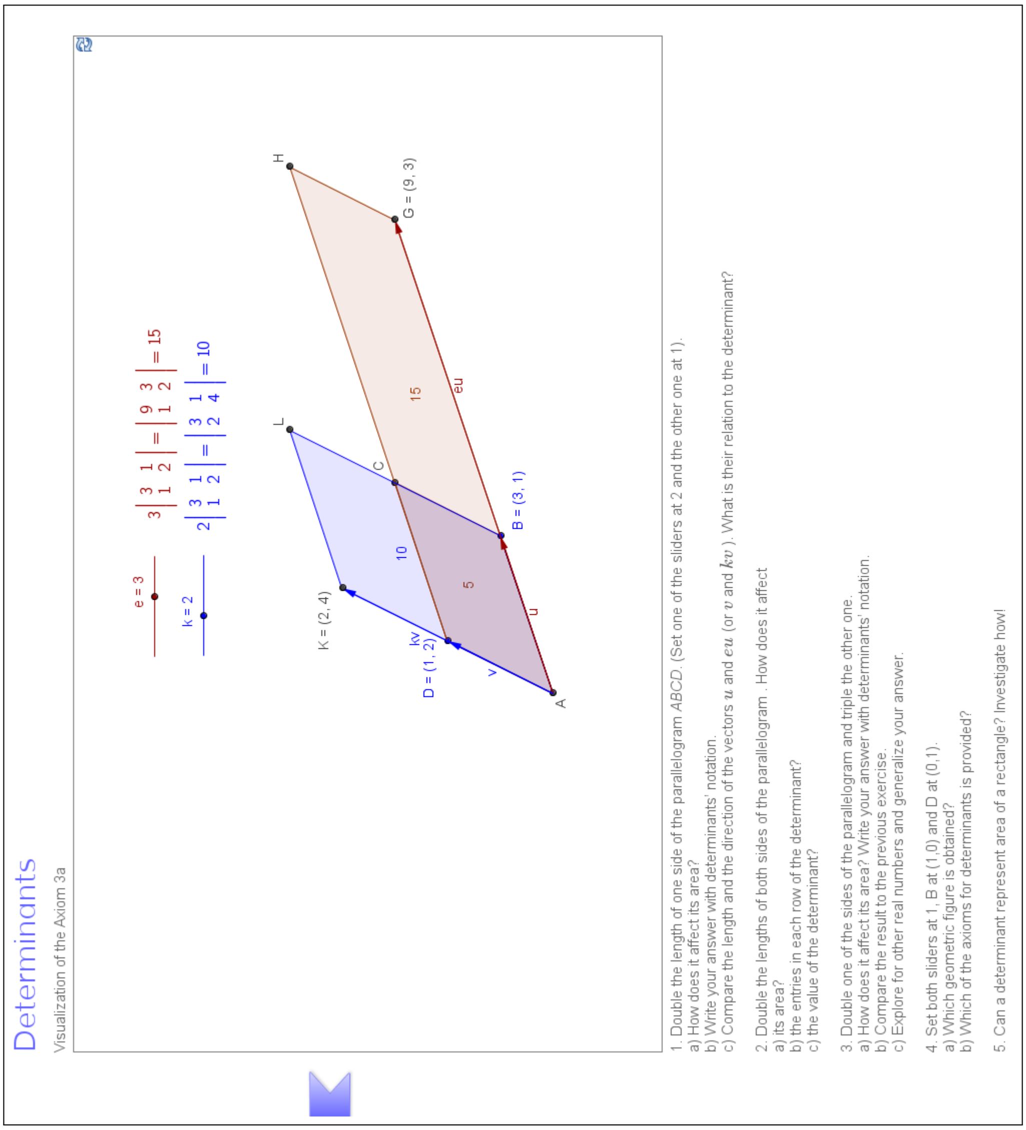


A.7 Additive Property of Dot Product of Vectors

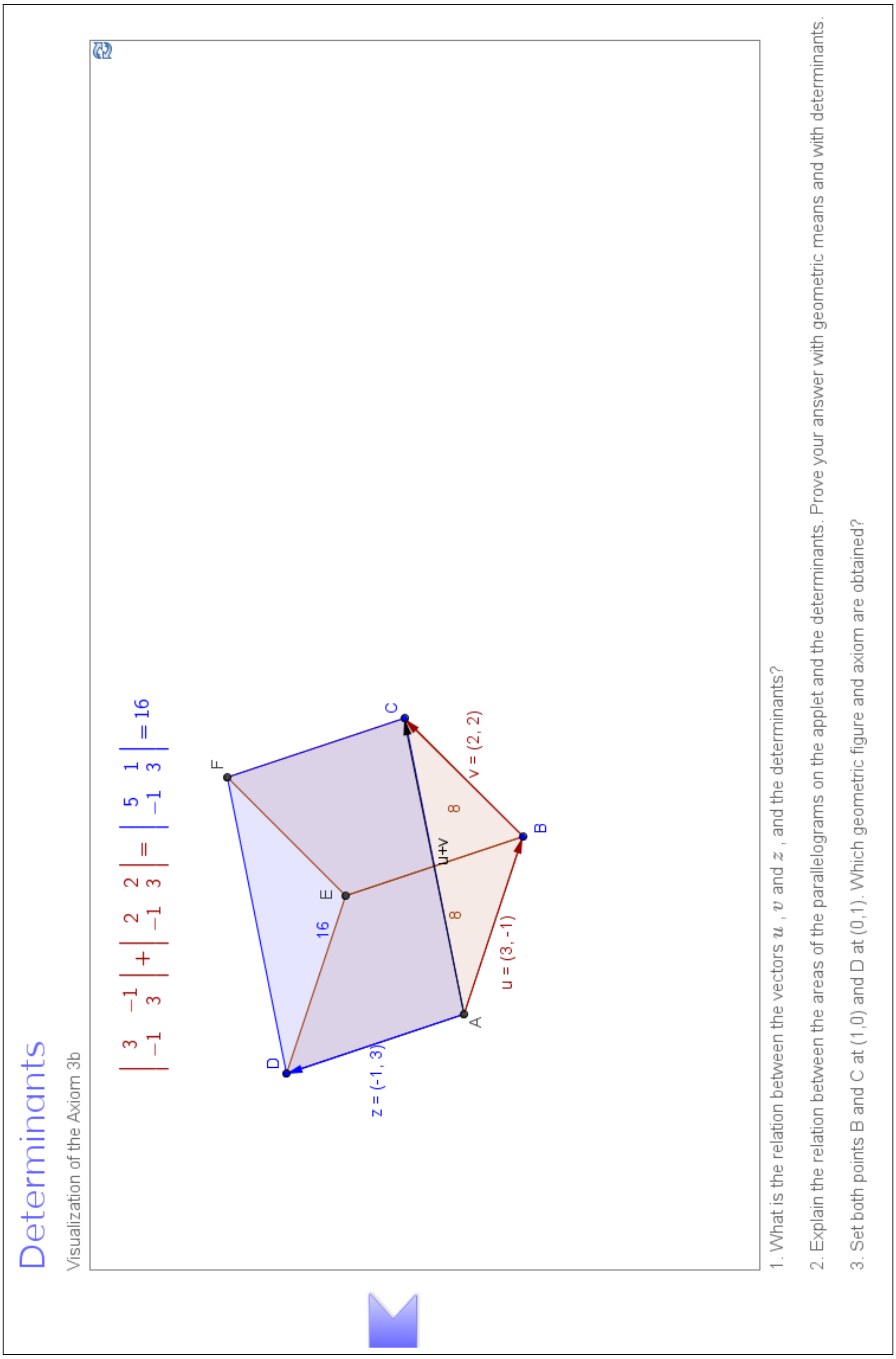


A.8 Chapter 3. Determinants

A.9 Scaling Property of Determinants



A.10 Additive Property of Determinants



Appendix B

Mathematics Dictionary

Mathematics Dictionary - Mathematisches Wörterbuch
Englisch-Deutsch: Basic Terms-Grundlagen

absolute value	Absolutbetrag; Betrag	matrix	Matrix
addition	Addition	modulus	Betrag
angle	Winkel	multiplication	Multiplikation
area	Fläche	natural	natürliche Zahl
associative	assoziativ	natural number	natürliche Zahl
associative law	Assoziativgesetz	necessary	notwendig
axis	Achse	negative	negativ
basis	Basis	nonnegative	nichtnegativ
Cartesian	kartesisch	nonsingular	nichtsingulär
centre	Mittelpunkt	number	Zahl
circle	Kreis	odd	ungerade
coefficient	Koeffizient	order	Ordnung
column	Spalte	origin	Ursprung
commutative	kommutativ	parallel	parallel
commutative law	Kommutativgesetz	plane	Ebene
component	Komponente	point	Punkt
conclusion	Folgerung	polynomial	Polynom
congruence	Kongruenz	positive	positiv
congruent	kongruent	power (x to the . of n)	Potenz (n-te von x)
coordinate	Koordinate	prime number	Primzahl
coordinate system	Koordinatensystem	product	Produkt
corollary	Korollar	quadrant	Quadrant
decreasing	fallend	ray	Strahl
definition	Definition	real number	realle Zahl
degree	Grad	relation	Relation
denominator	Nenner	representative	Repräsentant
determinant	Determinante	rotation	Drehung
diagonal	diagonal; Diagonale	rotation angle	Drehwinkel
diagram	Diagramm	rotation axis	Drehachse
difference	Differenz	rotation plane	Drehebene
distance	Abstand	row	Zeile
distributive law	Distributivgesetz	scalar product	Skalarprodukt
dot product	Skalarprodukt	set	Menge
element	Element	solution	Lösung
entry	Eintrag	solution set	Lösungsmenge
equal	gleich	space	Raum
equation	Gleichung	square	Quadrat
equivalence	Äquivalenz	square root	Quadratwurzel
equivalent	äquivalent	straight line	Gerade
Euclidean	euklidisch	subset	Teilmenge
even	gerade	substitution	Substitution
figure	Ziffer	sum	Summe
function	Funktion	symmetric	symmetrisch
geometry	Geometrie	system of equations	Gleichungssystem
greater than	größer als	theorem	Satz; Theorem
identity	Identität	triangle	Dreieck
increasing	steigend; wachsend	triangle inequality	Dreiecksungleichung
inequality	Ungleichung	union	Vereinigung
integer	ganze Zahl	unique	eindeutig
intersection	Durchschnitt	unit circle	Einheitskreis
interval	Intervall	vector	Vektor
linear	linear	volume	Volumen
mapping	Abbildung	zero	Null

Englisch-Deutsch: Linear Algebra- Lineare Algebra

algebra	Algebra	perpendicular	senkrecht
column	Spalte	plane	Ebene
column vector	Spaltenvektor	point	Punkt
coordinate	Koordinate	projection	Projektion
coordinate system	Koordinatensystem	quadrant	Quadrant
cross product	Kreuzprodukt	row	Zeile
determinant	Determinante	row vector	Zeilenvektor
diagonal matrix	Diagonalmatrix	scalar	Skalar
dimension	Dimension	scalar product	Skalarprodukt
dot product	Skalarprodukt	set	Menge
element	Element	similar	ähnlich
entry	Eintrag	similarity	Ähnlichkeit
Euclidean	euklidisch	space	Raum
identity	Identität	square	Quadrat
identity element	neutrales Element	symmetric	symmetrisch
identity matrix	Einheitsmatrix	system of equations	Gleichungssystem
intersection	Durchschnitt	triangle	Dreieck
inverse	Inverse	triangular matrix, upper / lower	Dreiecksmatrix, obere / untere
linear combination	Linearkombination	unit	Einheit
linearly (in)dependent	linear (un)abhängig	unit matrix	Einheitsmatrix
lower triangular matrix	untere Dreiecksmatrix	unit vector	Einheitsvektor
matrix	Matrix	upper triangular matrix	obere Dreiecksmatrix
module	Modul	vector	Vektor
nonsingular	nichtsingulär	volume	Volumen
normal	Normale	zero element	Nullelement
order	Ordnung	zero matrix	Nullmatrix
origin	Ursprung	zero vector	Nullvektor
parallel	parallel		

Appendix C

Introductory Survey for Preliminary Study

INTRODUCTORY SURVEY

***Note*:** This Survey is anonymous. Please answer each of the questions. Please provide explanation of your answers. The data collected with this survey will be used only for scientific research purposes in the area of teaching and learning Linear algebra and Analytic geometry.

Question 1. What is a vector?

Alternative question 1: How would you explain what a vector is to one of your classmates? Use as many different ways as you can.

Question 2. What is a linear combination of vectors?

Alternative question 2: How would you explain what a linear combination of vectors is to one of your classmates?

Appendix D

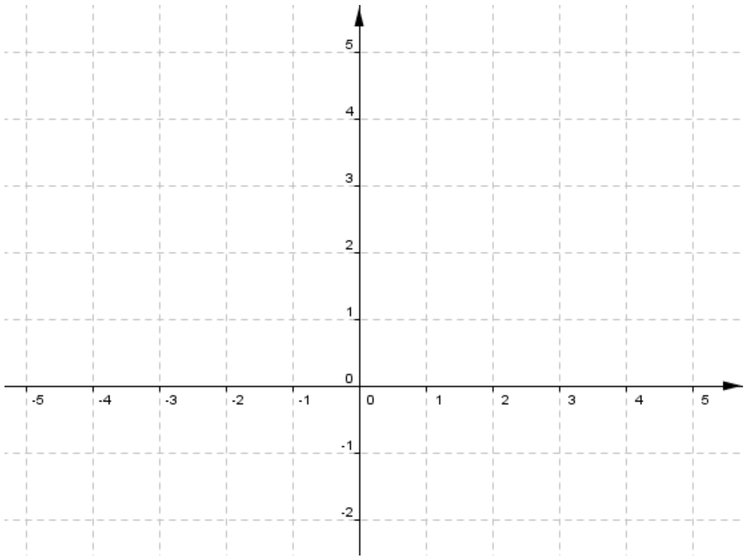
Worksheet. Tasks

Task 1. Parallelogram

a) Calculate and explain what is the geometric interpretation of the determinant $\begin{vmatrix} -4 & 4 \\ 3 & -1 \end{vmatrix}$.

b) Provide a geometric figure in the Cartesian coordinate system to validate your answer in a).

c) Prove!

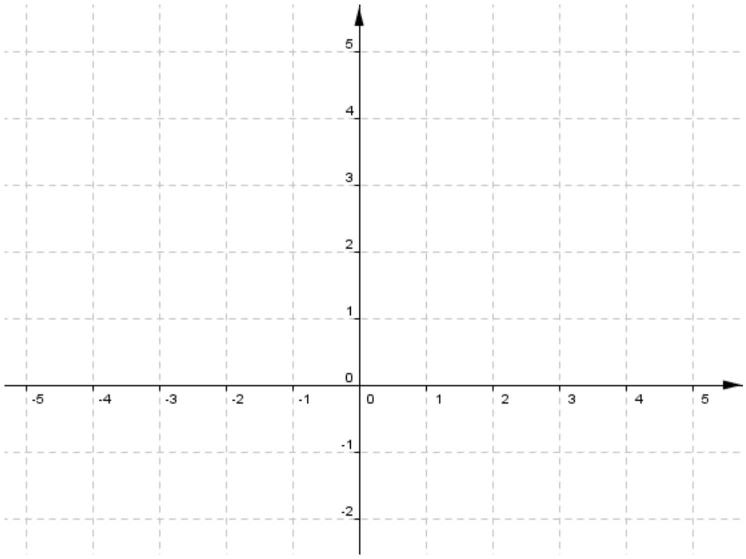


Task 2. Triangle

a) Sketch the triangle ABC if $A(5,0)$, $B(-1,4)$ and $C(-3,-2)$ in the Cartesian coordinate system.

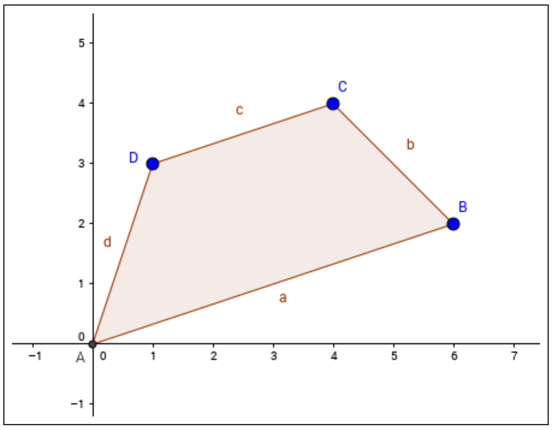
b) Calculate the area of the triangle ABC using determinants.

c) Write a formula for the area of a triangle ABC if $A(a,b)$, $B(c,d)$ and $C(e,f)$ using determinants.



Task 3. Trapezoid

Write the area of a trapezoid $ABCD$ given with coordinates of its vertices using determinants.



Task 4. Locus

Given the vertices $A(4, 0)$ and $B(8, 8)$ for a triangle ABC. Find the vertex C such that half the absolute value of the determinant is 24:

a) if $c \in y-$axis

b) if $c \in x-$axis

c) Which equations must all these vertices C satisfy?

d) What is the locus of the point C?

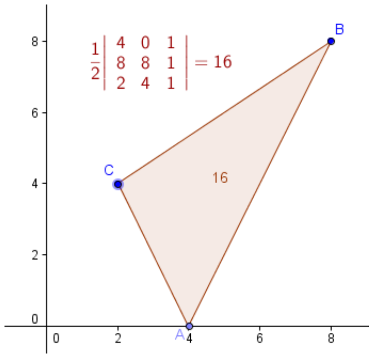


Figure. Initial Position of the Applet 4

Appendix E

Mathematical Journal

Standardized Students' Journal
Winter Semester 2012/ 13

Connecting and Visualizing Concepts in Linear Algebra and Analytic Geometry

School: ______________________________________

Student: ______________________________________

Class: __________________

Mentors:
Classteacher: Dr. Sabiene Zänker
Researcher: M.Sc. Ana Donevska-Todorova

Berlin, 2012

Date: _______________ Time: _______________ Code: _______________

Now, I am in a:
- good mood
- average mood
- bad mood

1. Todays' lesson was:

2. I have learned

3. I understand

4. I do not understand

5. My personal opinion (overview) on today's lesson, teaching method, examples, tasks, homework problems, applications etc.

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Acronyms

ALT	Actual Learning Trajectory
AG	Analytic Geometry
APOS	Action Process Object Scheme
ARQ	Auxiliary Research Question
CAS	Computer Algebra System
CRQ	Central Research Question
DGE	Dynamic Geometry Environment
DGS	Dynamic Geometry System
DBR	Design Based Research
HLT	Hypothetical Learning Trajectory
LA	Linear Algebra
SLE	System of Linear Equations
VDS	Variational Dragging Scheme

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Ana Donevska-Todorova
Berlin, den 24. März 2016